$G(2,4)$ and line geometry in $\mathbb{P}_3$.

PLÜCKER QUADRIC IN $\mathbb{P}_5$. Let V be a 4-dimensional vector space and $\mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$. Then

$$Q_P \stackrel{\text{def}}{=} \{\,\omega \in \Lambda^2 V \mid \omega \wedge \omega = 0\,\} \subset \mathbb{P}_5$$

is a non singular quadric called *a Plücker quadric*. Namely, consider a base $\{e_0, e_1, e_2, e_3\} \subset V$ and the induced base $\{e_{ij} = e_i \wedge e_j\} \subset \Lambda^2 V$. Let $\{x_{ij}\}$ be the homogeneous coordinates with respect to $\{e_{ij}\}$. Since $\left(\sum_{i<j} x_{ij} \cdot e_i \wedge e_j\right) \wedge \left(\sum_{i<j} x_{ij} \cdot e_i \wedge e_j\right) = 2\,(x_{01}x_{23} - x_{02}x_{13} + x_{03}x_{12}) \cdot e_0 \wedge e_1 \wedge e_2 \wedge e_3$, the Plücker quadric Q_P is presented by the quadratic equation

$$x_{02}x_{13} = x_{01}x_{23} + x_{03}x_{12}\ .$$

In fact, $\widetilde{q}(\omega_1, \omega_2) \stackrel{\text{def}}{=} \omega_1 \wedge \omega_2 \in \Lambda^4 V$ is a symmetric[1] bilinear form on $\Lambda^2 V$ defined up to proportionality, because $\Lambda^4 V = k$ as soon as a base vector is fixed in $\Lambda^4 V$; and Q_P is just the zero set of the corresponding quadratic form $q(\omega) = \widetilde{q}(\omega, \omega)$.

PLÜCKER EMBEDDING. By the definition, *a grassmannian* $G(2,4)$ is the set of all lines $\ell \subset \mathbb{P}_3 = \mathbb{P}(V)$, or equivalently, the set of all 2-dimensional vector subspaces $U \subset V$. *A Plücker map*

$$\mathrm{Gr}(2,4) \xhookrightarrow{\mathfrak{u}} \mathbb{P}(\Lambda^2 V)$$

sends a 2-dimensional subspace $U \subset V$ to the 1-dimensional subspace $\Lambda^2 U \subset \Lambda^2 V$. If U is spaned by a pair of vectors u_1, u_2 (that is, $\ell = \mathbb{P}(U)$ pass trough $u_1, u_2 \in \mathbb{P}(V)$), then up to proportionality $\mathfrak{u}(\ell) = \mathfrak{u}(U) = u_1 \wedge u_2$.

Lemma 10.1 *Two lines $\ell_1, \ell_2 \subset \mathbb{P}_3$ are intersecting iff* $\widetilde{q}(\mathfrak{u}(\ell_1), \mathfrak{u}(\ell_2)) = \mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell_2) = 0$.

Proof. Let $\ell_1 = \mathbb{P}(U_1)$, $\ell_2 = \mathbb{P}(U_2)$. If $U_1 \cap U_2 = 0$, then $V = U_1 \oplus U_2$ and there exist a base $\{e_i\} \subset V$ such that U_1 is spaned by e_0, e_1 and U_2 is spaned by e_2, e_3. So, $\mathfrak{u}(U_1) = e_0 \wedge e_1$, $\mathfrak{u}(U_2) = e_2 \wedge e_3$ and $\mathfrak{u}(U_1) \wedge \mathfrak{u}(U_2) = e_0 \wedge e_1 \wedge e_2 \wedge e_3 \neq 0$. If $U_1 \cap U_2 \neq 0$, then taking $u_0 \subset U_1 \cap U_2$ we can write $\mathfrak{u}(U_1) = u_0 \wedge u_1$, $\mathfrak{u}(U_2) = u_0 \wedge u_2$ for some u_1, u_2. So, $\mathfrak{u}(U_1) \wedge \mathfrak{u}(U_2) = u_0 \wedge u_1 \wedge u_0 \wedge u_2 = 0$. $\square$

Lemma 10.2 *If $\dim V = 4$, then $\omega \in \Lambda^2 V$ is decomposable*[2] *iff* $\omega \wedge \omega = 0$.

Proof. If ω is decomposable, say $\omega = u_1 \wedge u_2$, then $\omega \wedge \omega = u_1 \wedge u_2 \wedge u_1 \wedge u_2 = 0$, because of skew symmetry. To get the inverse, take a base $\{\xi_i\}$ such that ω turns into either $\omega = \xi_0 \wedge \xi_1 + \xi_2 \wedge \xi_3$ or $\omega = \xi_0 \wedge \xi_1$. If $\omega \wedge \omega = 0$, then the first case is impossible, because of $(\xi_0 \wedge \xi_1 + \xi_2 \wedge \xi_3) \wedge (\xi_0 \wedge \xi_1 + \xi_2 \wedge \xi_3) = 2\,\xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 \neq 0$. $\square$

Corollary 10.3 *The Plücker map is a bijection between the grassmannian $G(2,4)$ and the Plücker quadric $Q_P \subset \mathbb{P}_5$.*

Proof. $\mathfrak{u}$ is injective, because for any two lines $\ell_1 \neq \ell_2$ on $\mathbb{P}_3$ there exists a third line ℓ which intersect ℓ_1 and doesn't intersect ℓ_2. Then $\mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell) = 0$ and $\mathfrak{u}(\ell_2) \wedge \mathfrak{u}(\ell) \neq 0$, so, $\mathfrak{u}(\ell_1) \neq \mathfrak{u}(\ell_2)$. Surjectivity follows from Lemma 10.2. $\square$

Corollary 10.4 *For any point $p = \mathfrak{u}(\ell) \in Q_P$ the intersection $Q_P \cap T_p Q_P$ consists of all $\mathfrak{u}(\ell')$ such that $\ell \cap \ell' \neq \varnothing$.*

Proof. $T_p Q_P$ is a zero set of the linear form $\widetilde{q}(\mathfrak{u}(\ell), *)$. But by Lemma 10.1, $\widetilde{q}(\mathfrak{u}(\ell), \mathfrak{u}(\ell')) = 0 \iff \ell \cap \ell' \neq \varnothing$. $\square$

LINE NETS AND LINE PENCILS IN $\mathbb{P}_3$. A set of lines on $\mathbb{P}_3$ is called *a net* if it is represented by a plane $\pi \subset Q_P \subset \mathbb{P}_5$. If $\pi \subset Q_P$ is spaned by 3 non collinear points $p_i = \mathfrak{u}(\ell_i)$, $i = 1, 2, 3$, then

$$\pi = Q_P \cap T_{p_1} Q_P \cap T_{p_2} Q_P \cap T_{p_3} Q_P\ .$$

So, by Lemma 10.1 and Corollary 10.4 the corresponding line net consist of all lines which intersect 3 given pairwise intersecting lines. Hence, there are exactly two geometrically different line nets on $\mathbb{P}_3$:

α-net is a set of all lines passing through a given point $O \in \mathbb{P}_3$; the corresponding plain $\pi_\alpha(O) \subset Q_P$ is called *α-plane*. It is spanned by the Plücker images of any 3 non coplanar lines passing through O.

[1] $\omega_1 \wedge \omega_2 = \omega_2 \wedge \omega_1$ if at least one of ω_1, ω_2 has even degree

[2] recall that homogeneous polynomial is called decomposable if it is represented as a product of linear forms

β-net is a set of all lines laying on a given plane $\Pi \in \mathbb{P}_3$; the corresponding plain $\pi_\beta(\Pi) \subset Q_P$ is called *β-plane*. It is spanned by the Plücker images of any 3 lines which lay on Π and don't have a common intersection.

Note that any two planes of the same type have exactly 1-point intersection, namely,

$$\pi_\beta(\Pi_1) \cap \pi_\beta(\Pi_2) = \mathfrak{u}(\,\Pi_1 \cap \Pi_2\,) \qquad \text{and} \qquad \pi_\alpha(O_1) \cap \pi_\alpha(O_2) = \mathfrak{u}(\,(O_1O_2)\,)\;.$$

Two planes of different types, say $\pi_\beta(\Pi)$ and $\pi_\alpha(O)$, don't intersect each other if $O \notin \Pi$. If $O \in \Pi$, then $\pi_\beta(\Pi) \cap \pi_\alpha(O)$ is a pencil of lines $\ell \subset \mathbb{P}_3$ passing through O and laying on Π.

Exercise 10.5. Show that there are no other line pencils in $\mathbb{P}_3$, i. e. each line on $Q_P \subset \mathbb{P}_5$ has the form $\pi_\beta(\Pi) \cap \pi_\alpha(O)$ for some O and Π.

HINT. Consider the cone $C = Q_P \cap T_pQ_P$. It has a vertice at p and consists of all lines which pass through p and lay on Q_p. Fix a 3-dimensional hyperplane $H \subset T_pQ_P$ which doesn't contain p. Then $G = C \cap H$ is non singular quadric on H. So, any line passing through p has a form $(pp') = \pi_\alpha \cap \pi_\beta$, where $p' \in G$ and the planes π_α, π_β are spaned by p and two lines passing through p' on G (see fig. 10.1).

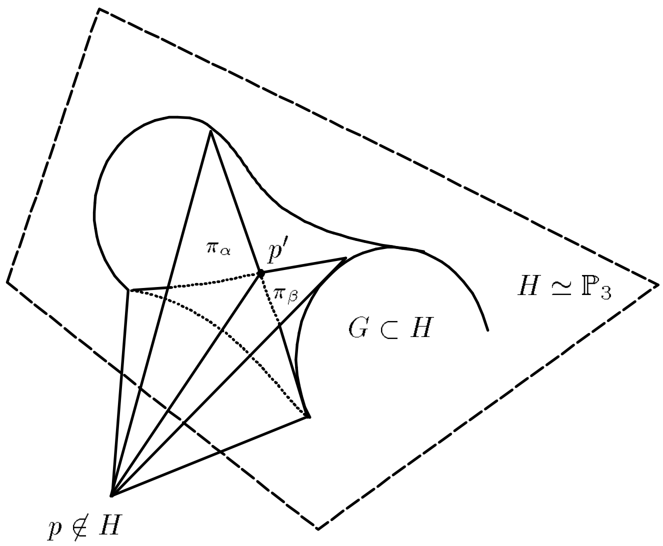

fig. 10.1. The cone $C = Q_P \cap T_pQ_P$.

<u>AFFINE CELL DECOMPOSITION OF $G(2,4)$.</u> Consider the following inclusion's diagram coming from the fig. 10.1:

$$\begin{tikzcd}
 & & \pi_\alpha \arrow[dr, hook] & & \\
p \arrow[r, hook] & \pi_\alpha \cap \pi_\beta \arrow[ur, hook] \arrow[dr, hook] & & C \arrow[r, hook] & Q_P \\
 & & \pi_\beta \arrow[ur, hook] & &
\end{tikzcd}$$

where the cone $C = Q_P \cap T_pQ_P$. If we take each stratum on this diagram and cut out all the smallest strata it contain, then we get a decomposition of $G(2,4)$ into disjoint union of cells which are identified[3] with the affine spaces:

$$\mathbb{A}^0 \sqcup \mathbb{A}^1 \sqcup \begin{matrix} \mathbb{A}^2 \\ \sqcup \\ \mathbb{A}^2 \end{matrix} \sqcup \mathbb{A}^3 \sqcup \mathbb{A}^4 \;=\; G(2,4)$$

Namely, $(\pi_\alpha \cap \pi_\beta) \setminus p \simeq \mathbb{A}^1$, because this is a projective line without a point. Then,

$$\pi_\alpha \setminus (\pi_\alpha \cap \pi_\beta) \simeq \pi_\beta \setminus (\pi_\alpha \cap \pi_\beta) \simeq \mathbb{A}^2\;,$$

because the both are projective planes without a line. Further, $C \setminus (\pi_\alpha \cup \pi_\beta) \simeq \mathbb{A}^1 \times \Big(G \setminus \big(G \cap T_{p'}G\big)\Big)$, just because C is a cone over G. Finally, $G \setminus \big(G \cap T_{p'}G\big) \simeq \mathbb{A}^2$ and $Q \setminus C \simeq \mathbb{A}^4$, because of the following lemma.

<u>Lemma 10.6</u> *Let $Q \subset \mathbb{P}_n$ be a quadric, $p \in Q$ be a non singular point, and $H \not\ni p$ be a codimension 1 hyperplane. Then the projection from p onto H induces a bijection between $Q \setminus (Q \cap T_pQ)$ and $\mathbb{A}^{n-1} = H \setminus (H \cap T_pQ)$.*

Proof. Any non tangent line passing through p have to intersect Q exactly one more time. All such lines are 1–1 parametrized by the points of $\mathbb{A}^{n-1} = H \setminus (H \cap T_pQ)$. $\square$

Exercise 10.7*. If you have some experience in topology, show that over $\mathbb{C}$ all odd integer homologies of $G(2,4)$ vanish and the even ones are $H_0 = H_2 = H_6 = H_8 = \mathbb{Z}$, $H_4 = \mathbb{Z} \oplus \mathbb{Z}$. Also, try to compute the homologies for the real grassmannian, where the boundary maps are non trivial.

[3] via a bijection given in coordinates by rational algebraic functions; in particular, for $k = \mathbb{R}, \mathbb{C}$ this identification is an homeomorphism