Grassmannians in general.

A GRASSMANNIAN $G(m,d)$ is defined as the set of all m-dimensional vector subspaces in a given d-dimensional vector space V. If the nature of V is important, we write $G(m,V)$ instead of $G(m,d)$. In the projective language, $G(m,d)$ is the set of all $(m-1)$-dimensional projective subspaces in $\mathbb{P}_{d-1}$. If $m=1$, then $G(m,d)=\mathbb{P}_{d-1}$. There is a canonical bijection $G(m,V)\simeq G(d-m,V^*)$ induced by duality. It sends $U\subset V$ to $\operatorname{Ann} U\subset V^*$ and wise versa.

Exercise 11.1. Let $\dim V=4$. Fix an isomorphism $V\xrightarrow{\widehat{q}}V^*$, say presented by a non singular quadric $Q\subset\mathbb{P}(V)$, and consider an automorphism of $G(2,V)$ given by $U\longmapsto\operatorname{Ann}\widehat{q}(U)$. Show that it maps the α-planes on $G(2,4)$ to the β-planes and wise versa.

PLÜCKER EMBEDDING $G(m,V)\xhookrightarrow{\mathfrak{u}}\mathbb{P}(\Lambda^m V)$ sends m-dimensional subspace $U\subset V$ to the 1-dimensional subspace $\Lambda^m U\subset\Lambda^m V$. If U is based by the vectors $\{u_1,u_2,\dots,u_m\}\subset U$, then $\mathfrak{u}(U)=u_1\wedge u_2\wedge\dots\wedge u_m$ up to proportionality, because taking an other base, say $v_i=\sum a_{ij}u_j$, we get

$$\mathfrak{u}(U)=v_1\wedge v_2\wedge\dots\wedge v_m=\det(a_{ij})\cdot u_1\wedge u_2\wedge\dots\wedge u_m\ .$$

Lemma 11.2 *The Plücker embedding is really injective.*

Proof. If $U_1\neq U_2$, then there exist a base in V such that some vectors $w_1,w_2,\dots,w_r$ of this base give a base for $U_1\cap U_2$, some other $u_1,u_2,\dots,u_{m-r}$ together with $\{w_\nu\}$ give a base for U_1, some other $v_1,v_2,\dots,v_{m-r}$ together with $\{w_\nu\}$ give a base for U_2, and the rest $e_1,e_2,\dots,e_{d+r-2m}$ are complementary to U_1+U_2. Let $\omega\in\Lambda^{d-m}V$ be the skew product of all v_ν and e_ν. The skew multiplication by ω $\Lambda^m V\xrightarrow{\xi\longmapsto\xi\wedge\omega}\Lambda^d V\simeq k$ turns into a linear form on $\Lambda^m V$ as soon as a base vector for $\Lambda^d V$ is fixed. This linear form does vanish at $\mathfrak{u}(U_2)$ and doesn't at $\mathfrak{u}(U_1)$. □

Example 11.3: 2×2-minors of 2×4-matrices. Let $\dim V=4$ and a base $\{e_1,e_2,e_3,e_4\}\subset V$ be fixed. Then a subspace $U\subset V$ based by u_1,u_2 can be presented as 2×4-matrix $A=\begin{pmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\end{pmatrix}$ whose rows are the coordinates of u_1,u_2. This matrix is defined by U up to the left multiplication $A\mapsto C\cdot A$ by any $C\in GL_2(k)$ (this corresponds to a base change in U). The Plücker embedding sends A to

$$u_1\wedge u_2=\sum_{i<j}\det\begin{pmatrix}a_{1i}&a_{1j}\\a_{2i}&a_{2j}\end{pmatrix}e_i\wedge e_j\ .$$

So, the homogeneous coordinates of $\mathfrak{u}(U)\in\mathbb{P}_5$ in the base $\{e_{ij}=e_i\wedge e_j\}$ are six 2×2-minors of A. In particular, the left multiplications by $C\in GL_2$ doesn't effect on the ratios between 2×2-minors of A. An other claim: six numbers $x_1,x_2,\dots,x_6$ give a collection of 2×2-minors for some 2×4 matrix iff they satisfy (maybe, after some renumbering) the Plücker equation $x_1x_2=x_3x_4+x_5x_6$.

MATRIX NOTATIONS AND PLÜCKER COORDINATES ON $G(m,d)$. A point $U\in G(m,d)$ can be presented by $(m\times d)$-matrix A_U whose rows are the coordinates of some base vectors $\{u_1,u_2,\dots,u_m\}\subset U$ with respect to a fixed base $\{e_1,e_2,\dots,e_d\}\subset V$. Such a matrix is not unique and is defined by U only up to the left multiplication by any $C\in GL_m$ (this corresponds to a base change in U). So, the grassmannian $G(m,d)$ can be considered as a factor space of $Mat_{m\times d}(k)$ by the left action of $GL_m(k)$. Under the Plücker embedding , the homogeneous coordinates of $\mathfrak{u}(U)\in\Lambda^m V$ in the standard base $e_I=e_{i_1}\wedge e_{i_2}\wedge\dots\wedge e_{i_m}$ are equal to the maximal minors of A. They are stable under the left GL_m-action and are called *Plücker coordinates* of U.

AFFINE COVERING AND AFFINE COORDINATES ON $G(m,d)$. Consider the standard affine card $\mathcal{U}_I\subset\mathbb{P}(\Lambda^m V)$ given by $x_I=1$, where x_I is the coordinate along $e_I=e_{i_1}\wedge e_{i_2}\wedge\dots\wedge e_{i_m}$. The inverse image $\mathfrak{U}_I\overset{\text{def}}{=}\mathfrak{u}^{-1}(\mathcal{U}_I)\subset G(m,d)$ consists of all U such that A_U has non zero maximal minor in the columns $(i_1,i_2,\dots,i_m)$. Any such U has *a unique* matrix representation $A_U^{(I)}=\left(a_{ij}^{(I)}\right)$ with *the identity $m\times m$-submatrix* staying in these columns. It is given by $A_U^{(I)}=A_{U,I}^{-1}\cdot A_U$, where A_U is an arbitrary matrix representation for U and $A_{U,I}\subset A_U$ is $m\times m$-submatrix formed by the columns $(i_1,i_2,\dots,i_m)$. So, the points of $\mathfrak{U}_I\subset G(m,d)$ are 1–1 parametrized by $m(d-m)$ matrix elements $\left(a_{\mu\nu}^{(I)}\right)$ staying outside the columns $(i_1,i_2,\dots,i_m)$ in $A_U^{(I)}$. In other words, we have *an affine chart* $\mathbb{A}^{m(d-m)}\xrightarrow{\simeq}\mathfrak{U}_I\subset G(m,d)$

which covers an open dense subset of the grassmannian. The charts $\mathfrak{U}_I$ are called *standard* and cover the whole of $G(m,d)$ when I runs through the length m increasing subsets in $\{1,2,\dots,d\}$.

Exercise 11.4. Write down the explicit *transition functions* between the standard affine charts $\mathfrak{U}_{12}$ and $\mathfrak{U}_{23}$ on $G(2,4)$.

Exercise 11.5*. If you had deal with smooth topology, check that real and complex grassmannians are the smooth (moreover, analytic) manifolds.

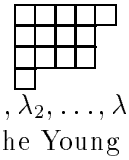
CELL DECOMPOSITION. The Gauss method shows that for any $U \subset V$ there exists *a unique* base $\{u_1,u_2,\dots,u_m\} \subset V$ such that the corresponding matrix $A_U = (a_{\mu\nu})$ is *a strong step matrix*, that is, $(a_{\mu\nu})$ contains the identity $m \times m$-submatrix, say in columns $(j_1,j_2,\dots,j_m)$, such that each row vanishes at the left of the unity coming from this identity submatrix (i. e. for all $i = 1,2,\dots,m$ we have $a_{ij} = 0 \quad \forall\, j < j_i$).

Exercise 11.6. Prove that different strong step matrices give different subspaces in V.

So, there exist a bijection between $G(m,d)$ and the set of all strong step matrices. The last one splits into disjoint union of the affine spaces. Namely, all strong step matrices that contain the identity submatrix in the fixed columns $I = (i_1,i_2,\dots,i_m)$ have exactly

$$mn - m^2 - (i_1 - 1) - (i_2 - 2) - \cdots - (i_m - m) = \dim G(m,d) - \sum_{\nu=1}^{m}(i_\nu - \nu)$$

free entries to put there any numbers from k. Hence, topologically, $G(m,d)$ is a disjoint union of $\binom{d}{m}$ affine cells $\mathfrak{A}_I$ enumerated by length m increasing subsets $I \subset \{1,2,\dots,d\}$. The I-th cell is homeomorphic to the affine space and has codimension $\sum\limits_{\nu=1}^{m}(i_\nu - \nu)$ in $G(m,d)$.

YOUNG DIAGRAM NOTATIONS. Traditionally, the ν-th difference $(i_\nu - \nu)$ in a length m increasing subset $I \subset \{1,2,\dots,d\}$ is denoted by $\lambda_{m+1-\nu}$ in order to have *a partition* $(d-m) \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0$ instead of the increasing collection $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant d$. By the definition, a partition λ is just a not increasing collection of non negative integers $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_m)$; *a length* $\ell(\lambda)$ is a number of the last non zero element in λ; *a weight* $|\lambda| \stackrel{\text{def}}{=} \sum\limits_{\nu}\lambda_\nu$; and  – a Young diagram of λ – is just a `flushleft`'ed collection of cell rows whose lengths are $\lambda_1,\lambda_2,\dots,\lambda_m$ (our diagram corresponds to the partition $(5,4,4,1)$). In other words, the partition is just the Young diagram, its weight is the number of cells, and its length is the number of rows.

Exercise 11.7. Check that there is a bijection between the length m increasing subsets $I \subset \{1,2,\dots,d\}$ and the Young diagrams contained in the $m \times (d-m)$ - rectangle.

In terms of Young diagrams, the grassmannian $G(m,d)$ is decomposed into the disjoint union of affine cells enumerated by the Young diagrams contained in the $m \times (d-m)$ - rectangle. λ-th cell has codimension $|\lambda|$ and is isomorphic to $\mathbb{A}^{m(d-m)-|\lambda|}$. In particular there is a unique 1-point cell, which has codimension $m\,(d-m)$ and corresponds to the rectangle itself, and a unique open dense cell of codimension zero, which corresponds to the empty diagram and coincides with the standard affine chart $\mathfrak{U}_{\{1,2,\dots,m\}}$. A topological closure of λ-th affine cell is called *a Schubert cycle* and denoted by σ_λ.

Exercise 11.8. Check that 6 Schubert cycles on the Plücker quadric $G(2,4) \simeq Q_P \subset \mathbb{P}_5$ are: $\sigma_{00} = Q_P$; $\sigma_{22} = p = (0:0:0:0:0:1) \in \mathbb{P}_5$; $\sigma_{10} = Q_P \cap T_pQ_P$; $\sigma_{11} = \pi_\alpha(O)$, where $O = (0:0:0:1) \in \mathbb{P}_3$; $\sigma_{20} = \pi_\beta(\Pi)$, where $\Pi \subset \mathbb{P}_3$ is given by $x_0 = 0$; $\sigma_{21} = \pi_\alpha(O) \cap \pi_\beta(\Pi)$.

Remark for who studied the topology. Clearly, the Schubert cycles generate the homologies of $G(m,d)$. Moreover, for the complex grassmannian they form the base of $H_*(G(m,\mathbb{C}^d),\mathbb{Z})$ over $\mathbb{Z}$. It is a nice (but not simle) combinatorial problem, to express the (topological) intersections of the Schubert cycles in terms of the Schubert cycles. The corresponding technique is known as *a Schubert calculus* and is described in Griffits-Harris, Fulton-Harris and Macdonald. Roughly speaking, *the homology ring* $H_*(G(m,\mathbb{C}^d),\mathbb{Z})$ is isomorphic to the truncated ring of symmetric polynomials via sending the Schubert cycles σ_λ to the Schur polynomials s_λ.

Exercise 11.9*. Show that $\sigma_{10}\sigma_{21} = \sigma_{20}^2 = \sigma_{11}^2 = \sigma_{22}$, $\sigma_{20}\sigma_{11} = 0$, and $\sigma_{10}^2 = \sigma_{20} + \sigma_{11}$ in the integer homology ring of complex $G(2,4)$.

HINT. To calculate σ_{10}^2 realize σ_{10} as $\sigma_{1,0}(\ell) = Q_P \cap T_{u(\ell)}Q_P = \{\ell'' \subset \mathbb{P}_3 |\ \ell \cap \ell'' \neq \varnothing\}$. Then, taking two *intersecting* lines ℓ and ℓ' in $\mathbb{P}_3$ we get $\sigma_{10}(\ell) \cap \sigma_{10}(\ell') = \pi_\alpha(O) \cup \pi_\beta(\Pi)$, where $O = \ell \cap \ell'$ and Π is spaned by ℓ and ℓ'.