Contractions and polarizations.

(Warning: in this lecture always $\operatorname{char} k = 0$**)**

SYMMETRIZATION AND ALTERNATION. A symmetric group $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting items of the decomposable tensors: $\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_n) \stackrel{\text{def}}{=} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \quad \forall\, \sigma \in \mathfrak{S}_n$. Subspaces

$$\mathsf{ST}^n V = \{\, t \in V^{\otimes n} \mid \sigma(t) = t \quad \forall\, \sigma \in \mathfrak{S}_n \,\} \tag{14}$$

$$\mathsf{AT}^n V = \{\, t \in V^{\otimes n} \mid \sigma(t) = \operatorname{sgn}(\sigma)\cdot t \quad \forall\, \sigma \in \mathfrak{S}_n \,\} \tag{15}$$

are called *symmetric* and *skew symmetric tensor* spaces. A map $V^{\otimes n} \xrightarrow{\text{alt}_n} \mathsf{AT}^n(V)$ such that

$$t \longmapsto \operatorname{alt}_n(t) \stackrel{\text{def}}{=} \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\cdot\sigma(t) \tag{16}$$

is called *an alternation*. It leaves fixed the skew symmetric tensors and projects all the tensors onto the skew symmetric ones. Analogously, *a symmetrization map* $V^{\otimes n} \xrightarrow{\text{sym}_n} \mathsf{AT}^n(V)$ which sends

$$t \longmapsto \operatorname{sym}_n(t) \stackrel{\text{def}}{=} \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \sigma(t) \tag{17}$$

projects the whole of the tensor space linearly onto the symmetric tensor subspace.

POLARIZATION. The skew multiplication of vectors $V_1 \times V_2 \times \cdots \times V_n \longrightarrow \Lambda^n V$, which is multilinear, induces *a linear* epimorphism $V^{\otimes n} \xrightarrow{\mathfrak{a}_n} \Lambda^n V \ :\ v_1 \otimes v_2 \otimes \cdots \otimes v_n \longmapsto v_1 \wedge v_2 \wedge \cdots \wedge v_n$ which is called a skew multiplication too. It may be considered also as a factorization map, which projects $V^{\otimes n}$ onto $\Lambda^n V = V^{\otimes n}/\mathcal{I}^{(n)}_{\text{skew}}$, where $\mathcal{I}^{(n)}_{\text{skew}} = (\mathcal{I}_{\text{skew}} \cap V^{\otimes n})$ is n-th degree homogeneous component of the skew commutativity relations ideal (see Tensor Guide, Lec. 8).

Claim 12.1 *Restricting the skew multiplication onto skew symmetric tensors* $\mathsf{AT}^n V \xrightarrow{\mathfrak{a}_n} \Lambda^n V$ *we get an isomorphism. Its inverse* $\Lambda^n V \xrightarrow{\text{pl}} \mathsf{AT}^n \subset V^{\otimes n}$ *sends* $\omega = v_1 \wedge v_2 \wedge \cdots \wedge v_n$ *to*

$$\operatorname{pl}(\omega) \stackrel{\text{def}}{=} \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\cdot v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \tag{18}$$

Proof. A map $V_1 \times V_2 \times \cdots \times V_n \longrightarrow \mathsf{AT}^n V$ which sends $(v_1, v_2, \dots, v_n)$ to the tensor (18) is skew symmetric and multilinear. Hence, there exist a unique linear map $\Lambda^n V \xrightarrow{F} \mathsf{AT}^n V$ which sends $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ to (18) as well. The composition $\mathfrak{a}_n \circ F = \text{Id}$, because it leaves fixed all the products $v_1 \wedge v_2 \wedge \cdots \wedge v_n$, whose linear span is $\Lambda^n V$. The composition $F \circ \mathfrak{a}_n = \text{Id}$, because it leaves fixed all the tensors (18), which linearly generate $\operatorname{im}(\operatorname{alt}_n) = \mathsf{AT}^n V$. So, $\text{pl} = F$ is what we need. □

Analogously, multilinear symmetric multiplication of vectors $V_1 \times V_2 \times \cdots \times V_n \longrightarrow S^n V$ induces *a linear* symmetric multiplication map $V^{\otimes n} \xrightarrow{\mathfrak{s}_n} S^n V \ :\ v_1 \otimes v_2 \otimes \cdots \otimes v_n \longmapsto v_1 v_2 \cdots v_n$ which is just the factorization epimorphism $V^{\otimes n} \twoheadrightarrow \Lambda^n V = V^{\otimes n}/\mathcal{I}^{(n)}_{\text{sym}}$, where $\mathcal{I}^{(n)}_{\text{sym}} = (\mathcal{I}_{\text{sym}} \cap V^{\otimes n})$ is n-th degree homogeneous component of the commutativity relations ideal (see Tensor Guide, Lec. 8).

Exercise 12.2. Check that restricting $\mathfrak{a}_n$ onto symmetric tensors $\mathsf{ST}^n V \xrightarrow{\mathfrak{s}_n} S^n V$ we get an isomorphism whose inverse $S^n V \xrightarrow{\text{pl}} \mathsf{ST}^n \subset V^{\otimes n}$ sends $f = v_1 v_2 \cdots v_n$ to

$$\operatorname{pl}(f) \stackrel{\text{def}}{=} \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \tag{19}$$

Each of two lifts $S^n V \xhookrightarrow{\text{pl}} V^{\otimes n} \xhookleftarrow{\text{pl}} \Lambda^n V$ is called *a full polarization* of the (skew) polynomials. For example, polarizing the skew base monomials, we get

$$\operatorname{pl}(e_I) = \operatorname{pl}(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\cdot e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}} \otimes \cdots \otimes e_{i_{\sigma(n)}} \tag{20}$$

and polarizing the symmetric base monomials, we get

$$\mathrm{pl}\,(e^{m_1}e^{m_2}\,\cdots\,e^{m_d}) = \frac{m_1!\,m_2!\,\cdots\,m_d!}{n!}\cdot e_{[m_1,\dots,m_d]} \tag{21}$$

where the symmetric tensor $e_{[m_1,\dots,m_d]}\in V^{\otimes n}$ is the sum of all decomposable tensors containing m_1 items e_1, m_2 items e_2, … , m_d items e_d (and $\sum\limits_{\nu} m_\nu = n$, certainly).

Exercise 12.3. Check directly that the sums $\sum\limits_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}(\sigma)\cdot e_{i_{\sigma(1)}}\otimes e_{i_{\sigma(2)}}\otimes\,\cdots\,\otimes e_{i_{\sigma(n)}}$ give a base for AT^nV and the sums $e_{[m_1,\dots,m_d]}$ give a base for ST^nV over any field of arbitrary characteristic. Certain that if $\mathrm{char}\,k>0$, then these sums are annihilated by the (skew) multiplication and the polarizations of basic (skew) monomials are not well defined.

Corollary 12.4 *If* $\mathrm{char}\,k=0$*, then there are following direct sum decompositions:*

$$\mathcal{I}^{(n)}_{skew}\oplus\mathsf{AT}^nV = V^{\otimes n} = \mathsf{ST}^nV\oplus\mathcal{I}^{(n)}_{sym}$$

where the projection $V^{\otimes n}\twoheadrightarrow\mathsf{ST}^nV$ *along* $\mathcal{I}^{(n)}_{sym}$ *is given by the symmetrization map* (17) *and the projection* $V^{\otimes n}\twoheadrightarrow\mathsf{AT}^nV$ *along* $\mathcal{I}^{(n)}_{skew}$ *is given by the alternation map* (16).

Proof. In the commutative diagram

$$\begin{tikzcd}
\mathsf{AT}^nV & & \mathsf{ST}^nV \\
& V^{\otimes n} \arrow[ul, "\mathrm{alt}_n"'] \arrow[ur, "\mathrm{sym}_n"] \arrow[dl, "\mathfrak{a}_n"] \arrow[dr, "\mathfrak{s}_n"'] & \\
\Lambda^nV \arrow[uu, "\mathrm{pl}"] & & S^nV \arrow[uu, "\mathrm{pl}"']
\end{tikzcd} \tag{22}$$

the vertical maps are isomorphisms. For any $t\in V^{oxn}$ we have $t-\mathrm{alt}_n(t)=\frac{1}{n!}\sum(t-\mathrm{sgn}(\sigma)\,\sigma(t))\in\ker(\mathfrak{a}_n)=\mathcal{I}^{(n)}_{\text{skew}}$, because of $\mathfrak{a}_n(t)=\mathrm{sgn}(\sigma)\,\mathfrak{a}_n(\sigma(t))$. So, $\ker(\mathrm{alt}_N)=\mathcal{I}^{(n)}_{\text{skew}}$. The same occurs in the symmetric case as well. □

<u>CONTRACTIONS.</u> If $\mathrm{char}\,k=0$, then the full contraction between $V^{\otimes n}$ and $V^{*\otimes n}$ induces the duality between $\Lambda^n(V)$ and $\Lambda^n\,(V^*)$ as well as between $S^n(V)$ and $S^n\,(V^*)$. Namely, for any degree n homogeneous (skew) polynomials f in $e_i\in V$ and ξ in $x_i\in V^*$ we put $\langle\, f\,,\,\xi\,\rangle \overset{\text{def}}{=} \langle\,\mathrm{pl}(f)\,,\,\mathrm{pl}(\xi)\,\rangle$, where the right side is the full contraction defined in the Tensor Guide.

Exercise 12.5. Let $\{e_1,e_2,\dots,e_n\}\subset V$ and $\{x_1,x_2,\dots,x_n\}\subset V^*$ be dual bases. Check that:

$$\langle\, e_I\,,\,x_J\,\rangle = \begin{cases}1/n! & \text{, for } I=J\\ 0 & \text{, for } I\neq J\end{cases} \tag{23}$$

$$\left\langle e^{m_1}e^{m_2}\,\cdots\,e^{m_d}\,,\,x^{\ell_1}x^{\ell_2}\,\cdots\,x^{\ell_d}\right\rangle = \begin{cases}\frac{m_1!\,m_2!\,\cdots\,m_d!}{n!} & \text{, if } m_\nu=\ell_\nu\ \forall\nu\\ 0 & \text{, otherwise}\end{cases} \tag{24}$$

HINT. The full contraction between $e_{i_1}\otimes e_{i_2}\otimes\,\cdots\,\otimes e_{i_n}$ and $x_{j_1}\otimes x_{j_2}\otimes\,\cdots\,\otimes x_{j_n}$ equals either 1 (only if $i_\nu=j_\nu\ \forall\nu$) or 0 (otherwise).

<u>(SKEW) POLYNOMIALS AND (SKEW) SYMMETRIC MULTILINEAR FORMS.</u> A full polarization $\mathrm{pl}(f)$ of a (skew) polynomial in $x_i\in V^*$ can be considered as a multilinear form on V. It sends $(v_1,v_2,\dots,v_n)$ to the full contraction $\widetilde{f}(v_1,v_2,\dots,v_n)\overset{\text{def}}{=}\langle\,v_1\otimes v_2\otimes\,\cdots\,\otimes v_n\,,\,\mathrm{pl}(f)\,\rangle$.

Exercise 12.6. Check that for a symmetric quadratic form q the above pass $q\longmapsto\widetilde{q}$ gives exactly the polar bilinear form we had deal with in Lectures 4 – 6.

Claim 12.7 *Polarization identifies* S^nV^* *and* Λ^nV^* *with the spaces of all symmetric and skew symmetric multilinear forms in* n *arguments on* V*.*

Proof. Any multilinear form φ may be presented via full contraction $\varphi(v_1,v_2,\dots,v_n)=\langle\,v_1\otimes v_2\otimes\,\cdots\,\otimes v_n\,,\,\xi\,\rangle$ with some $\xi\in V^{*\otimes n}$. Since for all $\sigma\in\mathfrak{S}_n$, $t\in V^{\otimes n}$, $\xi\in V^{*\otimes n}$ obviously $\langle\,\sigma(t)\,,\,\sigma(\xi)\,\rangle=\langle\,t\,,\,\xi\,\rangle$, we have $\langle\,\sigma(t)\,,\,\xi\,\rangle=\langle\,t\,,\,\sigma^{-1}(\xi)\,\rangle$ as well. So, φ has the same symmetry properties as ξ. □