Linear span of tensor and Plücker relations.

(Warning: in this lecture always $\operatorname{char} k = 0$)

LINEAR SPAN OF A TENSOR. Let $U, W \subset V$ be any two subspaces. Writing down the standard monomial bases, we see immediately that $(U \cap W)^{\otimes n} = U^{\otimes n} \cap W^{\otimes n}$ in $V^{\otimes n}$.

Exercise 13.1. Check this; also check that $S^n U \cap S^n W = S^n(U \cap U)$ in $S^n V$ and $\Lambda^n U \cap \Lambda^n W = \Lambda^n(U \cap U)$ in $\Lambda^n V$.

So, for any tensor $t \in V^{\otimes n}$ there is *a minimal* subspace *Span* $(t) \subset V$ whose n-th tensor power contains t. It is called *a linear span* of t and coincides with the intersection of all $W \subset V$ such that $t \in W^{\otimes n}$. Let us denote by $c_j^t : V^{*\otimes(n-1)} \longrightarrow V$ a linear map which sends a decomposable tensor $\xi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_{n-1}$ to its full contraction with t using $(n-1)$ indices of t except for the j-th index, that is, if

$$t = \sum_{(t_1, t_2, \ldots, t_n)} t_1 \otimes t_2 \otimes \cdots \otimes t_n \,, \quad \text{where } t_i \in V \,,$$

then $c_j^t(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_{n-1}) = \sum\limits_{t_j} \Big(\prod\limits_{\nu \neq j} \xi_\nu(t_\nu) \Big) \cdot t_j \in V$

Lemma 13.2 *Span* $(t) \subset V$ *is spaned by images of the contraction maps* $V^{*\otimes(n-1)} \xrightarrow{c_j^t} V$ *for* $j = 1, 2, \ldots, n$.

Proof. Let *Span* $(t) = W \subset V$. Then $t \in W^{oxn}$ and $\operatorname{im}\left(c_j^t\right) \subset W \; \forall j$. So, and it remains to prove that W is annihilated by any linear form $\xi \in V^*$ which annihilate all the subspaces $\operatorname{im}\left(c_j^t\right)$. To this aim, take such ξ as the first base vector for some base for V^* and let $e \in V$ be the first base vector of the dual base for V, i. e. $V = \operatorname{Ann}(\xi) \oplus e \cdot k$. Then, we can write $t = t' \otimes e + t''$, where $t' \in (\operatorname{Ann}\xi)^{\otimes(n-1)}$ and $t'' \in (\operatorname{Ann}\xi)^{\otimes n}$. Since $c_1^t\left(V^{*\otimes(n=1)}\right) \subset \operatorname{Ann}\xi$, for any $\zeta \in V^{*\otimes(n=1)}$ we have

$$0 = \xi(c_1^t(\zeta)) = \langle\, t \,,\, \zeta \otimes \xi \,\rangle = \langle\, t'' \,,\, \zeta \otimes \xi \,\rangle + \langle\, t' \otimes e \,,\, \zeta \otimes \xi \,\rangle = \langle\, t' \,,\, \zeta \,\rangle \,.$$

So, $t' \in V^{\otimes(n-1)}$ annihilates all $\zeta \in V^{*\otimes(n=1)}$. Hence, $t' = 0$. This means that in the decomposition

$$t = \sum_{(i_1, i_2, \ldots, i_n)} \tau_{(i_1, i_2, \ldots, i_n)} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \qquad (\text{with } \tau_{(i_1, i_2, \ldots, i_n)} \in k)$$

there are no terms with $e_{i_1} = e$. The same arguments for c_j^t with $j = 2, 3, \ldots, n$ show that for all $1 \leqslant j \leqslant n$ we have $e_{i_j} \neq e$ as well. So, $t \in (\operatorname{Ann}(\xi))^{\otimes n}$ and $W \subset \operatorname{Ann}(\xi)$. $\square$

DECOMPOSABLE SKEW SYMMETRIC POLYNOMIALS. Let $\omega \in \Lambda^n V$ and $\operatorname{pl}(\omega) \in V^{\otimes n}$ be its full polarization. Then, *Span* $(\operatorname{pl}(\omega)) \subset V$ coincides with the minimal $W \subset V$ such that $\omega \in \Lambda^n W$ (see diagram (22) from Lec. 12). For short, we denote this subspace by *Span* (ω) and call *a linear span* of ω as well. Since n contractions

$$V^{*\otimes(n-1)} \xrightarrow{c_j^{pl(\omega)}} V$$

are produced from each other by $\mathfrak{S}_n$-action on ω, *Span* (ω) coincides with the image of just one of them, because of skew symmetricity of $\operatorname{pl}(\omega)$. So, *Span* $(\omega) = c_n^{pl(\omega)}$ is linearly generated by the images $c_n^{pl(\omega)}\left(x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_{n-1}}\right)$, where $\{x_1, x_2, \ldots, x_d\}$ is a base of V^*. These images are calculated explicitly as follows. Let $\omega = \sum\limits_I \alpha_I e_I \in \Lambda^n V$. Consider $\operatorname{pl}(\omega) = \sum\limits_I \alpha_I \operatorname{pl}(e_I) \in \mathsf{AT}^n V$ as the skew symmetric multilinear form on $\underbrace{V^* \times V^* \times \cdots \times V^*}_{n}$. This form takes

$$(\zeta_1, \zeta_2, \ldots, \zeta_n) \longmapsto \widetilde{\omega}(\zeta_1, \zeta_2, \ldots, \zeta_n) \stackrel{\text{def}}{=} \big\langle \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n \,,\, \operatorname{pl}(\omega) \big\rangle$$

Then $c_n^{\operatorname{pl}(\omega)}(x_{j_1}, x_{j_2}, \ldots, x_{j_n}) \in V$ is the linear form $\zeta \longmapsto \widetilde{\omega}(\, x_{j_1}, x_{j_2}, \ldots, x_{j_{n-1}} \,,\, \zeta \,)$ on V^*. It vanishes, if x_{j_ν} are not pairwise different. Otherwise, its value at x_i is either zero (if i is one of j_ν) or $\dfrac{\operatorname{sgn}(\sigma)}{n!}\alpha_I$, where I is the increasing version of $\{ x_{j_1}, x_{j_2}, \dots, x_{j_{n-1}}, i\}$ obtained by the permutation $\sigma \in \mathfrak{S}_n$. So, $c_n^{\mathrm{pl}(\omega)}(x_{j_1}, x_{j_2}, \dots, x_{j_n})$ either is zero or up to a sign equals

$$v_J = \sum_{i \notin J} (-1)^{n-p(i,J)}\, \alpha_{J \sqcup i}\, e_i \tag{25}$$

where $J = (j_1, j_2, \dots, j_{n-1})$ and $p(i, J)$ is the number of place where i stays in the increasing permutation of $(x_{j_1}, x_{j_2}, \dots, x_{j_{n-1}}, i)$.

Claim 13.3 *Span* $\left(\sum\limits_I \alpha_I e_I\right)$ *is generated by* $\binom{d}{n-1}$ *the vectors* v_J *from* (25) *when* J *runs through all increasing length* $(n-1)$ *subsets in* $\{1, 2, \dots, d\}$. $\square$

Plücker relations. A skew polynomial $\omega = \sum\limits_I \alpha_I e_I \in \Lambda^n V$ is decomposable iff $v_J \wedge \omega = 0$ for all the vectors v_J from (25), because of the following lemma.

Lemma 13.4 $\omega \in \Lambda^n V$ *is decomposable* $\iff$ $\dim \mathit{Span}(\omega) = n$ $\iff$ $w \wedge \omega = 0 \;\; \forall\, w \in \mathit{Span}(\omega)$

Proof. The first equivalence and implication "$\Rightarrow$" in the second are completely trivial. So, it is sufficient to prove that if $\dim W > n$, then for any $\omega \in \Lambda^n W$ there exist $w \in W$ such that $w \wedge \omega \neq 0$ in $\Lambda^{n+1} W$. Fix a base $\{e_i\} \subset W$ and the corresponding base $\{e_I\} \subset \Lambda^n W$. Take a monomial e_I from ω and a base vector e_j with $j \notin I$. Certainly, $e_j \wedge \omega \neq 0$, because it contains a monomial $e_{j \cup I}$, which comes only as $e_j \wedge e_I$ and can not be eliminated. $\square$

Now, note that a base monomial $e_K \in \Lambda^{n+1} V$ appears in $\omega \wedge e_i$ iff $i \in K$, and has a coefficient $(-1)^{n+1-p(K,i)}\, \alpha_{K \setminus i}$, where $p(K, i)$ is the place number of i in K. So, e_K comes in $v_J \wedge \omega$ with a coefficient which up to a sign equals

$$\sum_{i \in K \setminus (K \cap J)} (-1)^{p(i,J,K)}\, \alpha_{J \sqcup i}\, \alpha_{K \setminus i} = 0 \,, \tag{26}$$

where $p(i, J, K)$ is the sum of the place numbers where i stays in the increasing version of $J \sqcup i$ and in K.

Claim 13.5 $\omega = \sum\limits_I \alpha_I e_I \in \Lambda^n V$ *is decomposable iff for any choice of increasing indices* $J = (j_1, j_2, \dots, j_{n-1})$ *and* $K = (k_1, k_2, \dots, k_{n+1})$ *the quadratic relations 26 hold.* $\square$

The relations 26 are known as *Plücker relations* and describe the image of Plücker embedding

$$G(n, V) \xhookrightarrow{\;\mathfrak{u}\;} \mathbb{P}(\Lambda^n V)\ .$$

Geometrically, they present the grassmannian $G(n, V)$ as an intersection of quadrics 26. Algebraically, they describe all collection of $\binom{d}{n}$ numbers α_I which give a full set of maximal minors for some $m \times d$ - matrix. Certainly, equations (26) are not "independent".

Exercise 13.6. Write down a number of Plücker relations for $G(2, 4) \subset \mathbb{P}_5$.