Partial derivatives and Veronese varieties.

(Warning: in this lecture always $\operatorname{char} k = 0$)

<u>From symmetric multilinear forms to polynomials.</u> As we have seen in the lecture 12, any polynomial $f \in S^nV^*$ can be lifted to a symmetric tensor $\mathrm{pl}(f) \in \mathsf{ST}^nV^*$ and considered as a symmetric multilinear form $\widetilde{f}(w_1, w_2, \dots, w_n) = \langle w_1 \otimes w_2 \otimes \cdots \otimes w_n \, , \, \mathrm{pl}(f)\rangle$ in n arguments on V; moreover, this procedure gives an isomorphism between polynomials and multilinear forms. If $\widetilde{f} = \varphi$, then the polynomial function f on V is recovered from φ as $f(v) = \varphi(v, v, \dots, v)$. Indeed, if $f = \xi_1\xi_2 \dots \xi_n$ is a product of linear forms, then

$$\widetilde{f}(v, v, \dots, v) = \left\langle v^{\otimes n} \, , \, \mathrm{sym}_n \left(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n\right) \right\rangle = \prod_{\nu} \xi_\nu(v) = f(v)$$

<u>Polarities.</u> For any vector $v \in V$ and any polynomial $f \in S^nV^*$ the contraction $\langle v \, , \, \mathrm{pl}(f) \rangle$ does not depend on the choice of index in $\mathrm{pl}(f)$ to be contracted, because the polarization $\mathrm{pl}(f) \in \mathsf{ST}^nV^*$ is symmetric. In terms of multilinear forms, $\langle v \, , \, \mathrm{pl}(f) \rangle$ is a symmetric multilinear form in $(n-1)$ arguments on V such that its value at $(w_1, w_2, \dots, w_{n-1})$ equals $\widetilde{f}(v \, , \, w_1, w_2, \dots, w_{n-1})$. A projection of $\langle v \, , \, \mathrm{pl}(f) \rangle$ from $\mathsf{ST}^{n-1}V^*$ to $S^{n-1}V^*$ is called *a polar polynomial* for f, v and is denoted by $\mathrm{pl}_v f$. Its value at $w \in V$ equals

$$\mathrm{pl}_v(w) = \widetilde{f}(v, w, w, \dots, w) \, .$$

The polar map $(v, f) \longmapsto \mathrm{pl}_v f$ is bilinear in v, f and $\mathrm{pl}_v\mathrm{pl}_w f = \mathrm{pl}_w\mathrm{pl}_v f$ for any $v, w \in V$. So, multiple polars $\mathrm{pl}_{\,v_1,v_2,\dots,v_m} \, f \overset{\text{def}}{=} \mathrm{pl}_{v_1}\mathrm{pl}_{v_2} \cdots \mathrm{pl}_{v_1} f$ are symmetric and multilinear in v_i and linear in f. Hence, they come from the linear map $S^mV \otimes S^nV \longrightarrow S^{n-m}V^*$, which sends $v_1v_2 \cdots v_m \otimes f$ to the polynomial

$$\mathrm{pl}_{\,v_1,v_2,\dots,v_m} \, f(w) = \widetilde{f}(\, v_1, v_2, \dots, v_m \, , w, w, \dots , w)$$

This map is called *a contraction*. The full contraction from the lecture 12 appears here for $m = n$.

<u>Lemma 14.1</u> *If some base $x_1, x_2, \dots, x_d$ in V^* is fixed and $v_i = x_i(v)$ are the coordinates of a vector $v \in V$, then*

$$\deg(f) \cdot \mathrm{pl}_v f = \frac{\partial f}{\partial v} \overset{\text{def}}{=} \sum_{i=1}^{d} v_i \, \frac{\partial f}{\partial x_i}$$

Proof. Let $\{e_i\}$ be base of V dual to $\{x_i\}$. By the bilinearity $\mathrm{pl}_v f$ and indices symmetry reasons, we have only to check that

$$\mathrm{pl}_{e_1} x^{m_1}x^{m_2} \cdots x^{m_d} = \frac{1}{n}\, \frac{\partial}{\partial x_1} \, x^{m_1}x^{m_2} \cdots x^{m_d}$$

for all monomials with $\sum m_\nu = n = \deg(f)$. As we know,

$$\mathrm{pl}\left(x^{m_1}x^{m_2} \cdots x^{m_d}\right) = \frac{m_1!\, m_2! \cdots m_d!}{n!} \cdot x_{[m_1,\dots,m_d]}$$

where $x_{[m_1,\dots,m_d]} \in \mathsf{ST}^nV^*$ is the sum of all different tensor products containing m_1 items x_1, m_2 items x_2, … , m_d items x_d. The contraction with e_1, say in the first index, kills all terms which do not have e_1 as the first item and removes the first e_1 from all other terms. Hence,

$$\left\langle e_1 \, , \, x_{[m_1,\dots,m_d]} \right\rangle = x_{[(m_1-1),m_2,m_3,\dots,m_d]} \, .$$

So, $\left\langle e_1 \, , \, pl\left(x^{m_1}x^{m_2} \cdots x^{m_d}\right) \right\rangle = \dfrac{m_1}{n}\, \mathrm{pl}\left(x_1^{m_1-1}x_2^{m_2}x_3^{m_3} \cdots x_d^{m_d}\right) = \dfrac{1}{n}\, \dfrac{\partial}{\partial x_1}\, x^{m_1}x^{m_2} \cdots x^{m_d}$. □

<u>Some properties of partial derivatives</u> The lemma shows that $\dfrac{\partial f}{\partial v}$ does not depend on a base choice and $m!\,\dfrac{\partial^m f}{\partial v^m}(w) = n!\widetilde{f}(\underbrace{v, v, \dots, v}_{m}, \underbrace{w, w, \dots, w}_{n}) = (n-m)!\,\dfrac{\partial^{n-m} f}{\partial w^{n-m}}(v)$ is bihomogeneous of degree $(m, n-m)$ in (v, w). Also, we get *a Taylor formula*

$$f(v + w) = \sum_{\nu=0}^{\deg f} \frac{1}{\nu!}\, \frac{\partial^\nu f}{\partial w^\nu}(v) \, ,$$

for any polynomial f over any field of zero characteristic. Indeed, we can assume f to be homogeneous, say of degree n; then $f(v+w)$ is the full contraction of $f \in S^nV^*$ with

$$(v+w)^m = \sum_m \binom{n}{m} v^m w^{n-m} \in S^nV$$

i. e. the Taylor formula is nothing more than Newton's binomial expansion written in partial derivatives notations.

<u>LINEAR SPAN OF A POLYNOMIAL.</u> A linear span *Span* (f) of a symmetric polynomial $f \in S^nV^*$ is a minimal subspace $W \subset V^*$ such that $f \in S^nW$. Clearly, it coincides with a linear span of the full polarization $\mathrm{pl}(f) \in \mathsf{ST}^nV^*$. So, by Lemma 13.2 from the previous lecture, *Span* (f) is an image of the contraction map $S^{n-1}V \otimes S^nV^* \longrightarrow V^*$. In terms of partial derivatives this looks as

<u>Corollary 14.2</u> *Span* (f) *is spanned by the linear forms*

$$\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_{n-1}}} f(x) \tag{27}$$

obtained from f *by all possible* $(n-1)$*-fold differenciations.* □

<u>VERONESE VARIETY IN $\mathbb{P}_N = \mathbb{P}(S^nV^*)$</u>, where $N = \binom{n}{d} - 1$ and $\dim V = d+1$, consists of pure n-th powers of linear forms, that is, of $f(x) = \xi^n(x)$, where $\xi \in \mathbb{P}(V^*)$. It has a rational parametrization given by the Veronese map

$$\mathbb{P}(V^*) \xhookrightarrow{\xi \longmapsto \xi^n} \mathbb{P}(S^nV^*)$$

The previous lemma presents the Veronese variety by a system of quadratic equations. Namely, $f(x) = \xi^n(x)$ for some $\xi \in V^*$ iff *Span* (f) is 1-dimensional subspace generated by ξ. So, f is pure n-th power iff any two linear forms (27) are proportional to each other. If we arrange the coefficients of two linear forms as the rows of some $2 \times (d+1)$-matrix, then their proportionality means that all 2×2-minors of this matrix vanish.

<u>Example 14.3:</u> Veronese's curve Let $\dim U = 2$, $\mathbb{P}_1 = \mathbb{P}(U^*)$, $\mathbb{P}_n = \mathbb{P}(S^nU^*)$, $\{t_0, t_1\}$ be a base of U^*, and

$$\binom{n}{i} t_0^i t_1^{n-i} \qquad \text{for} \quad 0 \leqslant i \leqslant n$$

be the corresponding base of S^nV^*. Then the Veronese embedding $\mathbb{P}_1 \hookrightarrow \mathbb{P}_n$ sends a linear form $(\alpha_0 t_0 + \alpha_1 t_1)$ to

$$(\alpha_0 t_0 + \alpha_1 t_1)^n = \sum_i \alpha_0^i \alpha_1^{n-i} \cdot \binom{n}{i} t_0^i t_1^{n-i}$$

and its image is a rational curve with the parametric equations $a_i = \alpha_0^i \alpha_1^{n-i}$, where a_i are the homogeneous coordinates in $\mathbb{P}_n$. If $f(t) = \sum_i a_i \cdot \binom{n}{i} t_0^i t_1^{n-i}$, then the linear forms (27) are exhausted by

$$\frac{\partial^i}{\partial t_0^i} \frac{\partial^{n-1-i}}{\partial t_1^{i-1-n}} f(t) = a_i t_0 + a_{i+1} t_1 \quad , \quad \text{where} \quad 0 \leqslant i \leqslant (n-1) \, .$$

So, f lies on the Veronese curve iff $\det \begin{pmatrix} a_i & a_{i+1} \\ a_{j-1} & a_j \end{pmatrix} = a_i a_j - a_{i+1} a_{j-1} = 0$ for all $0 \leqslant i < j \leqslant n$. This also can be written as $\mathrm{rk} \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_n \end{pmatrix} = 1$. For $n = 2$ we get the Veronese quadric $a_0 a_2 = a_1^2$ in $\mathbb{P}_2$. For $n = 3$ we have a rational cubic curve, which is called *a twisted cubic*, given as an intersection of 3 quadrics $a_0 a_2 = a_1^2$, $a_1 a_3 = a_2^2$, $a_0 a_3 = a_1 a_2$.

Exercise 14.4. Draw the picture and discover that the first two quadrics are simple cones with vertices at $(0 \colon 0 \colon 0 \colon 1)$ and at $(1 \colon 0 \colon 0 \colon 0)$; the third quadric is Segre's one and is ruled by two line families; two cones have a common line element $a_1 = a_2 = 0$, which joins the vertices but does not lie on the Segre quadric; the line element $a_0 = a_1 = 0$ of the first cone and the one $a_2 = a_3 = 0$ of the second do lie on the Segre quadric in the same ruling family. So, any two of 3 quadrics are intersected along the twisted cubic and one more line, that is, the Veronese curve can not be given by use of 2 equations only!