Geometry of Veronese curve.

(Warning: in this lecture we are working over $\mathbb{C}$ (or any algebraically closed field with char $= 0$.**)**

In this lecture we use the notations from Example 14.3, namely, $C_n \subset \mathbb{P}_n = \mathbb{P}(S^nU^*)$ is the Veronese curve, which consists of all polynomials $f(t) = (\alpha_0 t_0 + \alpha_1 t_1)^n$. The polynomials $\sum\limits_i a_i \cdot \binom{n}{i} t_0^i t_1^{n-i} \in S^nU^*$ and the linear forms $\alpha_0 t_0 + \alpha_1 t_1 \in \mathbb{P}(U^*)$ are coordinated by $(a_0 : a_1 : \ \cdots : a_n)$ and $(\alpha_0 : \alpha_1)$.

VERONESE'S CURVE IS TWISTED RATIONAL CURVE OF DEGREE n. C_n has a bijective parametric presentation $a_i = \alpha_0^i \alpha_1^{n-i}$ by homogeneous degree n polynomials. Its intersection with any hyperplane $\sum A_i a_i = 0$ consists of all $a(\alpha)$ that satisfy the *non zero* n-th degree equation[1] $\sum_i A_i \alpha_0^i \alpha_1^{n-i} = 0$. So, a hyperplane section of Veronese's curve always consists of n points counted with appropriate multiplicities, typically, of n distinct points. In particular, the linear span of any $(n+1)$ Veronese's curve points gives the whole of $\mathbb{P}_n$.

Exercise 15.1. Let $p_0, p_1, \ldots, p_n \in S^nU^*$ be any linearly independent polynomials. Show that the parametrized curve $\alpha \longmapsto (p_0(\alpha) : p_1(\alpha) : \ \ldots \ p_n(\alpha))$ turns into Veronese's curve under an appropriate linear automorphism of $\mathbb{P}_n$.

HINT. a matrix of the linear transformation in question is inverse to the matrix whose rows are the coefficients of p_i.

INSCRIBED SIMPLICES. For $1 \leqslant i \leqslant n$ let $p_i = \xi_i^n$ be any n distinct points on the Veronese curve $C_n \subset \mathbb{P}_n = \mathbb{P}(S^nU^*)$. For each i consider the pencil of hyperplanes passing through $(n-2)$-dimensional face $(p_1, \ldots p_{i-1}, p_{i+1}, \ldots, p_n)$ opposite to p_i in the $(n-1)$ dimensional simplex $(p_1, p_2, \ldots, p_n)$. These n pencils are parametrized uniformly by the points of $\mathbb{P}_1^{\times} = \mathbb{P}(U)$ as follows. For any $\xi \in \mathbb{P}(U^*)$ denote by $\widehat{\xi} \in \mathbb{P}(U)$ the annihilator[2] $\mathrm{Ann}\,(\xi)$ and for each i take the product $\tau_i = \widehat{\xi}_1, \cdots, \widehat{\xi}_{i-1}\widehat{\xi}_{i+1}, \cdots, \widehat{\xi}_1 \in S^{n-1}U$. Define a plane $\pi_i(u) \subset \mathbb{P}(S^nU^*)$, which corresponds to $u \in \mathbb{P}(U)$ in i-th pencil, as the annihilator of $u\tau_i \in S^nU$. This means that

$$\pi_i(u) = \left\{ f(t) \in S^nU^* \mid \widetilde{f}(\widehat{\xi}_1, \ldots, \widehat{\xi}_{i-1}, u, \widehat{\xi}_{i+1}, \ldots, \widehat{\xi}_n) = 0 \right\}$$

where $\widetilde{f}$ is the full polarization of f considered as a symmetric multilinear form on U. In particular, for $f(t) = \zeta^n(t) \in C_n$ we have

$$(\zeta^n)^{\sim}(\widehat{\xi}_1, \ldots, \widehat{\xi}_{i-1}, u, \widehat{\xi}_{i+1}, \ldots, \widehat{\xi}_n) = \zeta(u) \prod_{\nu \neq i} \zeta\left(\widehat{\xi}_\nu\right)$$

So, for any $u \in \mathbb{P}(U)$ the plane $\pi_i(u)$ pass through all $p_\nu = \xi_\nu^n$ with $\nu \neq i$ and through the point $p = \xi^n \in C_n$ whose ξ annihilate u (i. e. such that $u = \widehat{\xi}$). In other words,

$$C_n = \bigcup_u \pi_1(u) \cap \pi_2(u) \cap \cdots \cap \pi_n(u) \tag{28}$$

Since a linear isomorphism between projective lines is uniquely defined by the images of any 3 distinct points, we get the following corollary.

Claim 15.2 *The Veronese curve is uniquely recovered from any its $(n+3)$ distinct points $a, b, c, p_1, p_2, \ldots, p_n$ as follows. Consider n hyperplane pencils through the $(n-2)$-dimensional faces of the inscribed simplex $(p_1, p_2, \ldots, p_n)$ and parametrize them uniformly by $u \in \mathbb{P}_1$ in such a way that the hyperplanes passing through a, b, c appear in each pencil when $u = 0,\ 1,\ \infty$. Then C_n coincides with the incidence graph (28), where u runs through the parameter line $\mathbb{P}_1$.* □

PROJECTIVE LINEAR GROUP ACTION. The group $PGL_2(k)$ acts on $\mathbb{P}(S^nU^*)$ via linear variable changes $(t_0, t_1) \longmapsto (at_0 + bt_1, ct_0 + dt_1)$. Projective automorphisms $\mathbb{P}(S^nU^*) \xrightarrow{\sim} \mathbb{P}(S^nU^*)$ which are indused by these substitutions send the Veronese curve to itself. We call them *parameter changes* on Veronese's curve.

[1] it is nothing than general n-th degree polynomial!

[2] if $\xi = \alpha_0 t_0 + \alpha_1 t_1$, then $\widehat{\xi} = \alpha_1 e_0 - \alpha_0 e_1$, where $\{e_o, e_1\} \subset U$ is the base of U dual to $\{t_0, t_1\} \subset U^*$

Claim 15.3 *Let $p_1, p_2, \dots, p_n, a, b, c \in \mathbb{P}_n = \mathbb{P}(S^n U^*)$ be any $n+3$ points with no $(n+1)$ on the same hyperplane. Then there exists a projective linear isomorphism $\mathbb{P}_n \xrightarrow{\sim} \mathbb{P}_n$ which sends these points onto Veronese's curve C_n. This isomorphism is unique up to parameter changes on Veronese's curve.*

Proof. For each $i = 1, 2, \dots, n$ identify $\mathbb{P}_1 = \mathbb{P}(U)$ with a pencil of hyperplanes through $p_1, \dots p_{i-1}, p_{i+1}, \dots, p_n$ by sending $u = e_0, e_1, (e_1 - e_0)$ to the hyperplanes that contain a, b, c and denote by $\pi_i(u)$ the u-th hyperplane in the i-pencil. Let the hyperplane $(p_1, p_2, \dots, p_n)$ appear in i-th pencil when $u = u_i$. Then $u_1, u_2, \dots, u_n \in \mathbb{P}(U)$ are mutually distinct by the following reason. Consider 2-dimensional plane $\Pi = (a, b, c)$ and denote by q_1 and q_2 its intersection with i-th and j-th $(n-2)$-dimensional faces of $(p_1, p_2, \dots, p_n)$.

Exercise 15.4. Show that no 3 of q_1, q_2, a, b, c are collinear (in particular, q_1, q_2 are distinct).

HINT. In the contrary case some $(n+1)$ of $p_1, p_2, \dots, p_n, a, b, c$ would lie on a hyperplane

So, sitting on Π, we have the picture[3] shown on fig. 15.1. The i-th and the j-th hyperplane pencils are represented there by the line pencils through q_1 and q_2. The hyperplane $(p_1, p_2, \dots, p_n)$ corresponds to the line $(q_1 q_2)$. The parametr line $\mathbb{P}(U)$ is identified with the line pencil through $O = (aq_1) \cap (bp_2)$ in two ways: one sends u to the line through $\pi_i(u) \cap (bc)$ and the other sends u to the line through $\pi_j(u) \cap (ac)$. As we know, these parametrisations coincide with each other, because they attach the same u to the lines through a, b, c. The lines which correspond to $u = u_i$ and $u = u_j$ join O with $\pi_i(u_i) \cap (bc) = (q_1 q_2) \cap (bc)$ and with $\pi_j(u_j) \cap (ac) = (q_1 q_2) \cap (ac)$. Since O, q_1 and q_2 are not collinear, these 2 lines are distinct, i. e. $u_i \neq u_j$.

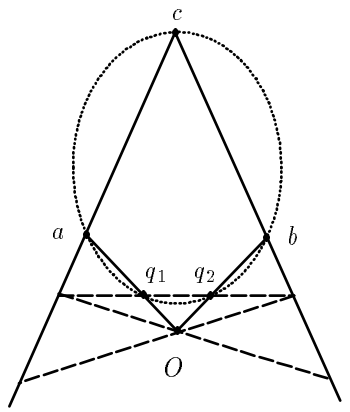

fig. 15.1. Comparing u_i and u_j.

Now denote by $\mathcal{G}$ the incidence graph (28) build from our current hyperplane pencils. Let $u_i = \mathrm{Ann}\,(\xi_i)$ for $\xi_1, \xi_2, \dots, \xi_n \in U^*$. By Lemma 3.3, there exists a unique projective linear automorphism $\mathbb{P}_n \xrightarrow{\sim} \mathbb{P}_n$ which sends $p_i \longmapsto \xi_i^n$ for $1 \leqslant i \leqslant n$, $a \longmapsto t_1^n$, and $b \longmapsto t_0^n$. It identifies $\mathcal{G}$ the with the Veronese curve, because it sends the hyperplane pencil through $p_1, \dots p_{i-1}, p_{i+1}, \dots, p_n$ to the one through $\xi_1^n, \dots \xi_{i-1}^n, \xi_{i+1}^n, \xi_n^n$ in such a way that $\pi_i(u)$ goes to the hyperplane through ξ^n as soon as $\xi = \mathrm{Ann}\, u$. Indeed, this takes place for $u = e_1$, $u = e_0$ and $u = u_i$ when a, b and p_i go to t_0^n, t_1^n and ξ_i^n. Hence, this holds for each u and for $u = e_1 - e_0$ we have $c \longmapsto (t_0 + t_1)^n$. That is the existence.

Uniqueness follows from the above construction as well. Namely, after appropriate parameter change we can suppose that the isomorphism in question sends a, b, c to t_0^n, t_1^n and $(t_0 + t_1)^n$. So, it induces the uniform parametrization of hyperplane pencils trough p_ν and this parametrization coincides with the above one. So, the images of $p_1, p_2, \dots, p_n$ are uniquely recovered as $\xi_i = \mathrm{Ann}\,(u_i)$. □

PROJECTING TWISTED CUBIC. Let us describe all plane projections of the twisted cubic $C_3 \supset \mathbb{P}_3 = \mathbb{P}(S^3 U^*)$. Up to a projective isomorphism, the projection does not depend on the choice of a target plane as soon as the center is fixed, because the projection of one target plane onto another gives an linear isomorphism between the projection images. Let $p = p(t) = \ell_1(t)\ell_2(t)\ell_3(t) \in \mathbb{P}_3 = \mathbb{P}(S^3 V^*)$ be a projection center. After some parameter change we can suppose that either $\ell_1 = \ell_2 = \ell_3 = t_0^2$, (this means that $p \in C$), or $\ell_1 = \ell_2 = t_0$, $\ell_3 = t_1$ [4], or $\ell_1 = (t_0 + t_1)$, $\ell_2 = (t_0 + \omega\, t_1)$, $\ell_3 = (t_0 + \omega^2 t_1)$, where $\omega = \sqrt[3]{1} \neq 1$ (that is, $p(t) = t_0^3 + t_1^3$ has 3 distinct roots).

In the first case $p = (1\colon 0\colon 0\colon 0)$; take a target plane to be $a_0 = 0$ with the coordinates $(x_0\colon x_1\colon x_2) = (a_1\colon a_2\colon a_3)$. Then the projection is given by parametric equations $(x_0\colon x_1\colon x_2) = (\alpha_0^2\colon \alpha_0\alpha_1\colon \alpha_1^2)$ and coincides with the plane Veronese conic $x_0 x_2 = x_1^2$.

In the second case $p = (0\colon 1\colon 0\colon 0)$; take a target plane to be $a_1 = 0$ with the coordinates $(x_0\colon x_1\colon x_2) = (a_0\colon a_2\colon a_3)$. Then the projection is given by parametric equations $(x_0\colon x_1\colon x_2) = (\alpha_0^3\colon \alpha_0\alpha_1^2\colon \alpha_1^3)$ and in the affine chart $\{x_0 = 1\}$ it turns into $x = \alpha^2$, $y = \alpha^3$, where $x = x_1/x_0$, $y = x_2/x_0$, and $\alpha = \alpha_1/\alpha_0$. So, we get a curve $y^2 = x^3$ or, in the homogeneous coordinates, $x_2^3 = x_3^2 x_0$. This curve is called *a cuspidal cubic*, because of the singularity form at the origin.

In the third case $p = (1\colon 0\colon 0\colon 1)$; take a target plane $\pi = \{a_2 = 0\}$ with the coordinates $(x_0\colon x_1\colon x_2) = ((a_0 - a_1)\colon a_1\colon a_2)$ (the first three coordinates w.r.t. the base $\{\, t_0^3, 3\, t_0^2 t_1, 3\, t_0 t_1^2, t_0^3 + t_1^3 \,\}$). So, the projection from $p = t_0^3 + t_1^3$ gives the parametrized curve $(x_0\colon x_1\colon x_2) = \big((\alpha_0^3 - \alpha_1^3)\colon \alpha_0^2\alpha_1\colon \alpha_0\alpha_1^2\big)$. In affine chart $x_0 = 1$ we get, like above, $x = \alpha/(1 - \alpha^3)$, $y = \alpha^2/(1 - \alpha^3)$ or $xy = x^3 - y^3$. This curve has a self intersection point at the origin and is called *a nodal* cubic.

[3] compare this construction with Remark 5.8 and drawings from the lecture 6

[4] geometrically, this means that p lies on a tangent line to C_3