Projective hypersurfaces.

(Warning: in this lecture we are working over $\mathbb{C}$ (or any algebraically closed field with char $= 0$.)

Within the next few lectures we use the following two facts about polynomials.

Fact 16.1 *The polynomial ring $k[x_1, \dots, x_n]$ is factorial*[1], *that is, any polynomial can be decomposed as $f = \prod p_i^{m_i}$, where p_i are mutually coprime irreducible polynomials, which are defined by f uniquely up to scalar factors.* □

Fact 16.2 *Let $X \subset \mathbb{A}_n$ be given by a polynomial equation $f(x) = 0$. If a polynomial g vanishes at any point of X, then g is divisible by any irreducible factor of f ([2]).*

Proof. This follows immediately from the factoriality of the polynomial ring and the famous Hilbert Nullstellensatz, which will be discussed (together with Fact 16.2) in Lect. 21. □

SPACE OF HYPERSURFACES. By the definition, a projective space $\mathbb{P}(S^d V^*)$ is called *a space of degree d hypersurfaces* in $\mathbb{P}_n = \mathbb{P}(V)$. It consists of all non zero d-th degree homogeneous polynomials considered up to proportionality. Its dimension is $\binom{d+n}{d}$. Its lines, planes and 3D-subspaces are called *pencils*, *nets* and *webs* of hypersurfaces. Geometrically, each polynomial whose prime factorization is $f(x) = \prod p_i(x)_i^m$ defines a zero set $(f)_0 = \{x \in \mathbb{P}_n \mid f(x) = 0\} = \bigcup m_i \cdot (p_i)_0$ — an union of (intersecting) *irreducible components* $(p_i)_0$ counted with formal integer multiplicities m_i. By Fact 16.2, each irreducible component $(p_i)_0$ does not admit any further splitting into an union $(g)_0 \cup (h)_0$ of proper subsurfaces.

INTERACTION WITH LINES. Let $S = \{F(x) = 0\}$ be a surface of degree d and $\ell = (pq) \in \mathbb{P}_n$ be a line spanned by $p, q \in V$, that is, $\ell = \mathbb{P}(\{\lambda p + \mu q\})$, where $\lambda, \mu \in k$. Consider $(\lambda : \mu)$ as homogeneous coordinates along ℓ. Then $\ell \cap S$ is given by the equation $f(\lambda, \mu) = 0$, where

$$f(\lambda, \mu) = F(\lambda p + \mu q) = \sum_{\nu=0}^{d} \lambda^\nu \mu^{n-\nu} \binom{d}{\nu} \widetilde{F}(p^\nu, q^{n-\mu}) \tag{29}$$

By the Newton - Taylor expansion, the coefficient $\widetilde{F}(p^\nu, q^{n-\mu})$ is bihomogeneous of degree $(\nu, n-\nu)$ in (p, q) and equals

$$\widetilde{F}(p^\nu, q^{n-\mu}) \overset{\text{def}}{=} \widetilde{F}(\underbrace{p, p, \dots, p}_{\nu}, \underbrace{q, q, \dots, q}_{d-\nu}) = \frac{\nu!}{d!} \frac{\partial^\nu F}{\partial p^\nu}(q) = \frac{(d-\nu)!}{d!} \frac{\partial^{d-\nu} F}{\partial q^{d-\nu}}(p)$$

where $\widetilde{F}$ is a unique symmetric multilinear form such that $\widetilde{F}(p, p, \dots, p) = F(p)$ ([3]). If $f(\lambda, \mu) \equiv 0$, i. e. $\widetilde{F}(p^\nu, q^{n-\mu}) = 0$ for all ν, then $\ell \subset S$. If $f(\lambda, \mu) \not\equiv 0$, then $\ell \cap S$ consists of d points counted with appropriate multiplicities. Namely, since k is algebraically closed, $f(\lambda, \mu) = \prod\limits_i (\alpha_i'' \mu - \alpha_i' \lambda)^{s_i}$ is a product of linear forms[4]. Each linear form corresponds to an intersection point $\alpha = (\alpha' : \alpha'') = \alpha' p + \alpha'' q \in \ell \cap S$. The maximal power s_i such that $f(\lambda, \mu)$ is divisible by $(\alpha_i'' \mu - \alpha_i' \lambda)^{s_i}$ is called *a local intersection index* between S and ℓ at α. It is denoted by $(S, \ell)_\alpha$. So, $\deg S = \sum\limits_{\alpha \in S \cap \ell} (S, \ell)_\alpha$ as soon as $\ell \not\subset S$.

TANGENT LINES AND TANGENT SPACE. A line ℓ is called *a tangent line* to S if there is a point $p \in S \cap \ell$ with $(S, \ell)_p \geqslant 2$. We say that ℓ does *touch* S at p.

Claim 16.3 *Let $p \in S = \{x \in \mathbb{P}_n \mid F(x) = 0\}$ and $q \in \mathbb{P}_n$ be any point. Then (pq) touch S at p iff $\widetilde{F}(p^{n-1}, q) = 0$.*

Proof. If $p \in S$, then affine version of (29) near p takes the form $F(p + tq) = t \binom{d}{1} \widetilde{F}(p^{n-1}, q) + t^2 \binom{d}{2} \widetilde{F}(p^{n-2}, q^2) + \cdots$ because of the first term $\widetilde{F}(p^n) = F(p) = 0$. Local intersection index of S and (pq) at p is the maximal power of t factored out of $F(p + tq)$. So, it is $\geqslant 2$ iff $\widetilde{F}(p^{n-1}, q) = 0$. □

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[1]Fact 16.1 is true over any ground field k; it follows from more general theorem, which says that for any commutative ring A without zero divisors $A[x]$ is factorial as soon as A is. You find the proof in any standard university algebra course (say, in van der Waerden's book). The key point is the following Gauss lemma: let A be a factorial ring with the quotient field K and $f \in A[x]$; then any factorization $f = gh$ in $K[x]$ induces a factorization $f = \tilde{g}\tilde{h}$ in $A[x]$ and the new factors have the same degrees as the old ones (in fact, this is just a redistribution of the coefficient denominators between the factors).

[2]Fact 16.2 holds over an algebraically closed field of any characteristic.

[3]i. e. $\widetilde{F} \in \mathsf{ST}^d V^*$ is the full polarization of F considered as a form on V

[4]namely, $f(\lambda, \mu) = \mu^d f(t, 1)$, where $t = \lambda/\mu$ and $f(t, 1) \in k[t]$; now, $f(t, 1) = \prod (t - \alpha_i)^{m_i}$

Corollary 16.4 *The union of all tangent lines through $p\in S$ is a projective space* $T_pS \overset{\text{def}}{=} \left\{ y\in\mathbb{P}_n \mid \sum_{i=0}^{n} y_i\,\frac{\partial F}{\partial x_i}(p)=0 \right\}$. *It is either a hyperplane or the whole of $\mathbb{P}_n$. The last happens iff* $\frac{\partial F}{\partial x_i}(p)=0\ \forall i$. □

The space T_pS is called *a tangent space* to S at p. If $T_pS=\mathbb{P}_n$, then S is called *singular* at p and p is called *a singular point* of S. Otherwise p is called *a smooth point* of S. S is called *smooth* if all its points are smooth.

Corollary 16.5 *Let q be either any point outside S or a smooth point on S. Then the apparent contour of S from q, that is, the set of all points $p\neq q$ where lines through q touch S is cut out by degree $(d-1)$ hypersurface* $S_q^{(d-1)} \overset{\text{def}}{=} \left\{ y\in\mathbb{P}_n \mid \sum_{i=0}^{n} q_i\,\frac{\partial F}{\partial x_i}(y)=0 \right\}$ *; in particular, the equation for $S_q^{(d-1)}$ is not identically zero.*

Proof. Indeed, $\frac{1}{d}\sum_{i=0}^{n} q_i\,\frac{\partial F}{\partial x_i}(y) = \mathrm{pl}_qF(y) = \widetilde{F}(y^{n-1},q)$. If it vanishes identically in y, then taking $y=q$ we get $F(q)=0$, i. e. $q\in S$, and $F(q,q,\dots,q,y)=\mathrm{pl}_q^{n-1}F(y)=\mathrm{pl}_q^{n-2}\mathrm{pl}_qF(y)$ vanishes identically in y all the more. Hence, q is singular point of S. So, under the corollary assumption, the touching condition at y $\widetilde{F}(y^{n-1},q)=0$ is non zero degree $(n-1)$ equation in y. □

Point multiplicities. A number $\mathrm{mult}_S(p) \overset{\text{def}}{=} \min_{\ell\ni p}(\ell,S)_p$ is called *a multiplicity* of p on S. If $p\in S$ is singular, then any line through p intersects S with index $\geqslant 2$ at p. So, p is smooth iff p has the multiplicity 1. A point p has multiplicity $\geqslant m$ iff all order $(m-1)$ partial derivatives of F vanish at p.

Polar hypersurfaces. A hypersurface $S_q^{(r)} \overset{\text{def}}{=} \left\{ y\in\mathbb{P}_n \mid \widetilde{F}(q^{n-r},y^r)=0 \right\}$ is called *a r-th degree polar* of S with respect to p. If $F(q^{n-r},y^r)$ vanishes identically in y, we say that the polar is *trivial*, that is, it coincides with the whole of $\mathbb{P}_n$. Intuitively, if $q\in S$ is a smooth point, then $S_q^{(r)}$ is a degree r tangent surface to S. Indeed, S and $S_q^{(r)}$ have the same polars of degrees $\leqslant r$ w.r.t. q. For example, the linear polar is just the tangent space, quadratic polar is a quadric through q with the same tangent space as S, and so on. If $q\in S$ is singular and has a multiplicity $m\geqslant 2$, then all the polars of degree $\leqslant (m-1)$ w. r. t. p are trivial and the m-th degree polar is non trivial but singular at p.

Example 16.6: The space of singular conics. Let V be 3D vector space, $\mathbb{P}_5=\mathbb{P}(S^2V^*)$ be the space of conics on $\mathbb{P}_2=\mathbb{P}(V)$, and $S\subset\mathbb{P}_5$ be a locus of the singular conics. Let us fix some coordinates and present quadratic forms $q(x)\in S^2V^*$ as $q(x)=x\cdot A\cdot {}^tx$ with symmetric 3×3-matrices A. Since q is singular iff $\det A=0$, we see that S is an irreducible cubic hypersurface in $\mathbb{P}_5$. We would like to find its singular points and describe non singular tangent hyperplanes. By Laplace's relations, $\det A=\sum_{\nu}(-1)^{i+\nu}a_{i\nu}A_{i\nu}$, where $A_{i\nu}$ is $(n-1)\times(n-1)$-minor situated outside i-th row and j-th column. So, $\frac{\partial \det A}{\partial a_{ij}}=(-1)^{i+j}A_{ij}$ and a point $q\in S_2$ is singular iff $\mathrm{rk}\,A=1$.

Exercise 16.7. Show that any rank 1 $m\times n$ matrix a_{ij} has $a_{ij}=\lambda_i\mu_j$ for some $\lambda_1,\lambda_2,\dots,\lambda_m,\ \mu_1,\mu_2,\dots,\mu_n\in k$.
Hint. A linear operator $k^n \xrightarrow{x\mapsto Ax} k^m$ has $\mathrm{rk}\,A=1$ iff $\dim\mathrm{im}\,A=1$; if $w=(\lambda_1,\lambda_2,\dots,\lambda_m)$ generates $\mathrm{im}\,A$, then $A(v)=\alpha(v)\,w$, where $k^n\xrightarrow{\alpha}k$ is linear form, say $\alpha=(\mu_1,\mu_2,\dots,\mu_n)$ …

In our case $A=(a_{ij})$ is symmetric and we should have $\mu_i=\lambda_i$, i. e. $a_{ij}=\lambda_i\lambda_j$ for some $\lambda_0,\lambda_1,\lambda_2\in k$. So, $A\in S$ is singular iff $q(x)=\left(\sum\lambda_ix_i\right)^2$ is a double line. That is, the set of singular points of S coincides with 2-dimensional Veronese's surface $\mathcal{V}\subset S$, which parametrizes double lines in $\mathbb{P}_2$. Now, let $q(x)=x\cdot A\cdot{}^tx$ be a smooth point of S, that is, a pair of distinct intersecting lines $\ell_1\cup\ell_2\subset\mathbb{P}_2$. In this case the correlation map $V\xrightarrow{\widehat{q}}V^*$ has 1-dimensional kernel spanned by the intersection point $v=\ell_1\cap\ell_2$. So, $\mathrm{cork}\,A=1$, not all A_{ij} are zeros, and the vertex v can be presented by any row or any column of the adjoint matrix $\widehat{A}=\left((-1)^{i+j}A_{ij}\right)$, because of the Laplace relations. In particular, $\mathrm{rk}\,\widehat{A}=1$, and $(-1)^{i+j}A_{ij}=\mu_i\mu_j$ for some $\mu_0,\mu_1,\mu_2\in k$, and $v=(\mu_0:\mu_1:\mu_2)$. So,

$$B=(b_{ij})\in T_qS_n \iff \sum_{ij} b_{ij}\cdot(-1)^{i+j}A_{ij}=0 \iff \sum_{ij} b_{ij}\,\mu_i\mu_j=0 \iff v\cdot B\cdot{}^tv=0$$

In other words, the tangent space T_qS at $q=\ell_1\cup\ell_2\subset\mathbb{P}_2$ consists of all conics passing through the point $\ell_1\cap\ell_2\in\mathbb{P}_2$.

Exercise 16.8. Let $\dim V=n+1$ and $S\subset\mathbb{P}(S^2V^*)$ be the space of singular quadrics on $\mathbb{P}_n=\mathbb{P}(V)$, and $q\subset\mathbb{P}_n$ be a simple cone quadric with the vertex at $p\in\mathbb{P}_n$. Show that q gives a non singular point of S and T_qS consists of all quadrics passing through p.