Plane curves: intersection indices.

(Warning: in this lecture we are working over $\mathbb{C}$ (or any algebraically closed field with char $=0$.)

ROOTS OF BINARY FORM. A polynomial $A(t) = a_0\,t_1^d + a_1\,t_0\,t_1^{d-1} + a_2\,t_0^2\,t_1^{d-2} + \cdots + a_{d-1}\,t_0^{d-1}\,t_1 + a_d\,t_0^d$ with zeros at $\alpha_1, \alpha_2, \ldots, \alpha_d$, where $\alpha_i = (\alpha_i' : \alpha_i'') \in \mathbb{P}_1$, is factorized as $A(t) = \prod_{i=0}^{d}(\alpha_i' t_1 - \alpha_i'' t_0)$. In particular, each a_i is bihomogeneous of degree $(i, d-i)$ in (α', α'') and equals $a_i = (-1)^{d-i}\sigma_m(\alpha', \alpha'')$, where

$$\sigma_i(\alpha', \alpha'') = \sum_{\#I=i}\Big(\prod_{i\in I}\alpha_i' \cdot \prod_{j\notin I}\alpha_j''\Big)$$

is a homogeneous version of the i-th elementary symmetric function (as usually, I runs through the increasing length i subsets in $\{1, 2, \ldots, d\}$).

RESULTANT. Fix two degrees $m, n \in \mathbb{N}$ and consider a polynomial ring $k[\alpha', \alpha'', \beta', \beta'']$ in four collections of variables $\alpha' = (\alpha_1', \alpha_2', \ldots, \alpha_n')$, $\alpha'' = (\alpha_1'', \alpha_2'', \ldots, \alpha_n'')$, $\beta' = (\beta_1', \beta_2', \ldots, \beta_m')$, $\beta'' = (\beta_1'', \beta_2'', \ldots, \beta_m'')$ The product

$$R_{AB} \overset{\text{def}}{=} \prod_{i,j}(\alpha_i'\beta_j'' - \alpha_i''\beta_j') = \prod_{i=1}^{n} B(\alpha_i) = (-1)^{mn}\prod_{j=1}^{m} A(\beta_j)\ ,$$

where $A(t) = \sum_{i=0}^{n} a_i\,t_0^i\,t_1^{n-i}$, $B(t) = \sum_{j=0}^{m} b_j\,t_0^j\,t_1^{m-j}$, $a_i = (-1)^i\sigma_i(\alpha', \alpha'')$, $b_j = (-1)^j\sigma_j(\beta', \beta'')$, is called *a resultant* of the polynomials $A(t)$, $B(t)$. Clearly, $R_{A,B}$ is bihomogeneous of bidegree (mn, mn) in (α, β) and may be expressed in terms of the coefficients of A, B.

Claim 17.1 *Up to a scalar factor, $R_{A.B}$ coincides with the Sylvester determinant*

$$det\ \underbrace{\begin{pmatrix} a_0 & a_1 & \ldots & a_n & & & \\ & a_0 & a_1 & \ldots & a_n & & \\ & & \ddots & \ddots & & \ddots & \\ & & & a_0 & a_1 & \ldots & a_n \\ b_0 & b_1 & \ldots & b_m & & & \\ & b_0 & b_1 & \ldots & b_m & & \\ & & \ddots & \ddots & & \ddots & \\ & & & b_0 & b_1 & \ldots & b_m \end{pmatrix}}_{m+n} \begin{array}{l} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \end{array}$$

Proof. Consider a vector space U with a basis $\{t_0, t_1\}$ and a linear map $S^{m-1}U \oplus S^{n-1}U \xrightarrow{M_{A,B}} S^{m+n-1}U$ which sends a pair of polynomials $(h_1(t), h_2(t))$ to $A(t)h_1(t) + B(t)h_2(t)$. The Sylvester matrix is transpose to the matrix of $M_{A,B}$ in the standard monomial bases. If a point (α, β) lies on the quadric $\alpha_i'\beta_j'' - \alpha_i''\beta_j' = 0$, then $(\alpha_i't_1 - \alpha_i''t_0) = (\beta_i't_1 - \beta_i''t_0)$ up to a scalar factor and this linear form divides $A(t)$, $B(t)$, and any polynomial $A(t)h_1(t) + B(t)h_2(t)$, that is, $\operatorname{im} M_{A,B} \neq S^{m+n-1}U$. So, the Sylvester determinant vanishes along each quadric $\alpha_i'\beta_j'' - \alpha_i''\beta_j' = 0$. Hence, it is divisible by $R_{A,B}$ and the quotient has to be a constant, because the both polynomials are of the same degree (mn, mn) in (α, β). $\square$

Example 17.2: Elimination technique. Let C_1, C_2 be two plane curves of degrees m and n given by equations $F(x) = 0$ and $G(x) = 0$ in $x = (x_0 : x_1 : x_2)$. Consider F, G as (non homogeneous) polynomials in x_0 with the coefficients in $k(x_1, x_2)$ and take their resultant[1] $R_{F,G}(x_1, x_2) \in k[x_1, x_2]$. If it is identically zero, then F and G have a common divisor in $k(x_1, x_2)[x_0]$. The Gauss lemma implies that this common divisor can be taken with the coefficients in $k[x_1, x_2]$. So, if $R \equiv 0$, then C_1 and C_2 have a common component. If $R \not\equiv 0$, then (x_1, x_2)-coordinates of any intersection point $p \in C_1 \cap C_2$ have to satisfy the resultant equation $R_{F,G}(x_1, x_2) = 0$ which is homogeneous of degree mn. So, the curves have either a common component or at most mn intersection points, which may be found by solving a homogeneous polynomial equation in x_1, x_2 only. These procedure[2] is called *an elimination of a variable*.

[1] the resultant of nonhomogeneous polynomials $F(x_0)$ and $G(x_0)$ is defined as the resultant of $t_0^n F(t_1/t_0)$ and $t_0^m G(t_1/t_0)$

[2] see a good example in Miles Ried's "Undergraduate AG" (LMS Student Texts 12, CUP (1988), p.103–106)

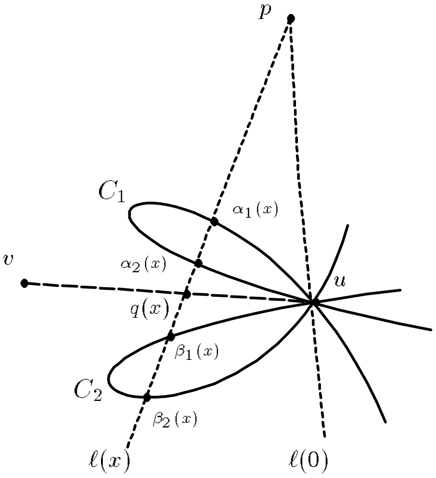

fig. 17.1. Zeuthen's rule.

<u>Zeuthen's rule.</u> Let C_1, C_2 be without common components and $u \in C_1 \cap C_2$. Fix two points $p, v \notin C_1 \cap C_2$ such that the line (pu) does not contain any intersection point of C_1, C_2 except for u. Let us parametrize the pencil of lines through p by the points of the line (uv) as $\ell(x) = (p, q(x))$, where $q(x) = u + x\,v$, $x \in k$, see fig. 17.1. When x varies, $C_1 \cap \ell(x) = \{\alpha_0(x), \alpha_1(x), \dots, \alpha_n(x)\}$ and $C_2 \cap \ell(x) = \{\beta_0(x), \beta_1(x), \dots, \beta_n(x)\}$ draw the branches of C_1 and C_2. In terms of $\ell(x)$, these point collections are the zeros of the polynomials $f_x(t_0, t_1) \stackrel{\text{def}}{=} F(t_0 p + t_1 q(x))$ and $g_x(t_0, t_1) \stackrel{\text{def}}{=} G(t_0 p + t_1 q(x))$ in t. Let $\beta_1(x), \beta_1(x), \dots, \beta_r(x)$ and $\alpha_1(x), \beta_2(x), \dots, \beta_s(x)$ be all the branches coming to u when $x \longrightarrow 0$. Define *a local intersection multiplicity* $(C_1, C_2)_u$ as the sum of orders of rs infinitesimals $\alpha_i(x) - \beta_j(x)$ with respect to $x \longrightarrow 0$ (this principle is known as *Zeuthen's rule*). Clearly, the sum of orders of $\alpha_i(x) - \beta_j(x)$ coincides with the order of R_{f_x, g_x} when $x \longrightarrow 0$.

<u>Lemma 17.3</u> *Consider $p = (p_0, p_1, p_2)$ and $q = (q_0, q_1, q_2)$ as independent variables and the resultant of $f(t_0, t_1) = F(t_0 p + t_1 q)$, $g(t_0, t_1) = G(t_0 p + t_1 q)$ as a polynomial $R_{f,g}(p, q) \in k[p, q]$. Then it is factorized as*

$$R_{f,g}(p,q) = const \cdot \prod_{w \in C_1 \cap C_2} det^{m_\xi} \begin{pmatrix} w_0 & w_1 & w_2 \\ p_0 & p_1 & p_2 \\ q_0 & q_1 & q_2 \end{pmatrix} \tag{30}$$

where the multiplicities m_u can be calculated by the Zeuthen rule applied to any pencil like on fig. 17.1.

Proof. Consider a prime factorization $R_{f,g}(p, q) = \prod Q_i^{r_i}(p, q)$ where Q_i are irreducible and mutually non proportional, and denote the determinants in the right side of (30) by $D_w(p, q)$. Geometrically, irreducible quadric $D_w(p, q) = 0$ consists of all pairs $p, q \in \mathbb{P}_2$ whose joining line (pq) belongs to the pencil of lines through w. If $w \in C_1 \cap C_2$, then $R(p, q)$ vanishes along the quadric $D_w(p, q) = 0$, that is, any such $D_w(p, q)$ divides the resultant. If for each i we point out some $D_w(p, q)$ in the right side of (30) such that no Q_ν with $\nu \neq i$ is divisible by it, then we get $Q_i(p, q) = D_w(p, q)$. We find such D_w as soon as we find a pair of distinct points $a, b \in \mathbb{P}_2$ such that $Q_i(a, b) = 0$, but $Q_\nu(a, b) \neq 0$ for $\nu \neq i$. Indeed, since such (a, b) annihilates the resultant, the line $(a, b) \subset \mathbb{P}_2$ passes through some $w \in C_1 \cap C_2$, that is, $D_w(a, b) = 0$; hence, this $D_w(p, q)$ does not divide any $Q_\nu(p, q)$ with $\nu \neq i$, because of $Q_\nu(a, b) \neq 0$. A point (a, b) we need exists, certainly, because, say, the product $(p_0 - q_0) \prod\limits_{\nu \neq i} Q_\nu(p, q)$ does not vanish everywhere along the hypersurface $Q_i = 0$. Indeed, if it does, then Q_i divides it. This implies that $Q_i(p, q) = p_0 - q_0$ and $R_{f,g} = 0$ along a line (a', b'), if $a'_0 = b'_0$, that is, along *any* line. But only lines which meet $C_1 \cap C_2$ do annihilate R.

To check the Zeuthen rule, take p and $q = q(x)$ as on the fig. 17.1 and note that only $D_w(x)$ with $w = u$ vanishes at $x = 0$ in (30), because of the choice of p, u, v; and this one $D_u(x) = \det(u, p, u + x\,v) = x \det(u, p, v)$ is proportional to t. $\square$

<u>Intersection theory of plane curves.</u> Lemma 17.3 shows that the local intersection multiplicities defined by the Zeuthen rule do not depend on the choice of p, v as soon as they are outside $C_1 \cap C_2$ and $(pu) \cap C_1 \cap C_2 = \{u\}$. Further, since $D_w(p, q)$ are bilinear in (p, q) and the resultant is bihomogeneous of degree (mn, mn), we get *the Bézout theorem*: $\sum\limits_{w \in C_1 \cap C_2} (C_1, C_2)_w = \deg C_1 \cdot \deg C_2$ for any two curves without common components.

It follows immediately from the Zeuthen rule, that the local intersection multiplicities are "distributive" with respect to the curve branches, that is, if C_1 has b_1 branches passing through u and C_2 has b_2 ones, then (C_1, C_2) is the sum of $b_1 b_2$ mutual intersection indices between the branches. In particular, if we denote by $d_1 C_1 + d_2 C_2$ a curve whose equation is $F^{d_1} G^{d_2} = 0$, then for any curves C'_1, C''_1 and any $d', d'' \in \mathbb{N}$ we have $(d' C'_1 + d'' C''_1, C_2)_u = d'\,(C'_1, C_2)_u + d''\,(C''_1, C_2)_u$.

<u>Affine localization.</u> If we restrict fig. 17.1 onto affine chart where (pv) is the infinity, (uv) is the x-axis, and (up) is the y-axis, then the line pencil through p turns into the line family $x = \text{const}$ and C_1, C_2 get affine equations $f(x, y) = 0$, $g(x, y) = 0$, which may be considered as (nonhomogeneous) polynomials in y with the coefficients in $k[x]$. Then the branches $y = \alpha_i(x)$ and $y = \beta_j(x)$ (that is, the roots of these polynomials) may be presented, say, as formal fractional power series in x (which convergent near $x = 0$, if $k = \mathbb{C}$). Sometimes this gives an easy way to compute local intersection multiplicity at the origin by looking either at the orders of $\alpha_i(x) - \beta_j(x)$ or at the order of the resultant $R_{f,g}(x)$.