Plane curves: singularities and tangents.

(Warning: in this lecture we are working over $\mathbb{C}$ (or any algebraically closed field with char $= 0$.)

GEOMETRICAL TANGENTS. Let $C \subset \mathbb{P}_2$ be a curve of degree d given by an equation $F(x) = 0$ and $p \in C$ be a point of multiplicity m, that is, all the polars $\dfrac{\partial^{d-i}F}{\partial p^i}(x)$ of degrees $0 \leqslant i \leqslant (m-1)$ vanish identically, but m-th degree polar $G(x) \stackrel{\text{def}}{=} (x) = F(\underbrace{p,p,\dots,p}_{d-m}, \underbrace{x,x,\dots,x}_{m})$ is non trivial. Since t-expansion $G(p+tq) = \sum_{\nu=1}^{m} t^\nu \binom{m}{\nu} \widetilde{F}(\underbrace{p,p,\dots,p}_{d-\nu}, \underbrace{q,q,\dots,q}_{\nu}) = t^m\, G(q)$ contains just one term, a line (p,q) either intersect the m-th degree polar only at p with multiplicity m or is a component of m-th degree polar. Hence, m-th degree polar is an union of lines $\ell_1, \ell_2, \dots, \ell_s$ called *geometrical tangent lines* to C at p. Algebraically, this leads to the decomposition $G(x) = \xi_1^{m_1}(x)\,\xi_2^{m_2}(x)\cdots\xi_s^{m_s}(x)$, where $\sum m_\nu = m$ and $\xi_1, \xi_2, \dots, \xi_s$ are mutually distinct linear forms whose zeros are the geometrical tangent lines $\ell_1, \ell_2, \dots, \ell_s$. In particular, if p is smooth, that is, $m = 1$, there is a unique tangent line $\ell = T_pC$ of multiplicity 1; a local intersection index $(\ell, C)_p \geqslant 2$ measures an order of the contact between ℓ and C.

Example 18.1: the simplest singularities. An m-typle point $p \in C$ is called *an m-typle node* (or *an m-typle selfintersection*) if there are m distinct geometrical tangents $\ell_1, \ell_2, \dots, \ell_m$ to C through p. Geometrically, this means that C has m mutually transversal branches through p. The difference $(\ell_i, C)_p - m - 1$ is called *an order of the contact* between ℓ_i and the corresponding branch of C. A node is called *ordinary* if all the geometrical tangents have the second order contacts with its branches, that is, $(\ell_i, C)_p = m + 1\ \forall\, i$.

A double point $p \in C$ is called *a cusp* (or *a selfcontact*) if the quadratic polar of p is a double line ℓ. Geometrically, this means that C has two branches which do touch each other at p. The unique geometrical tangent ℓ at p is called *a nodal tangency*. A local intersection number $(\ell, C)_p$ measures *an order of the selfcontact* for C at p; clearly, $(\ell, C)_p \geqslant 3$. If, A cusp is called *ordinary* if $(\ell, C)_p = 3$ is minimal possible.

We say that C has *only the simplest singularities* if the singular points of C are exhausted by ordinary double nodes and ordinary cusps. Two cubic curves with the simplest singularities are shown on the figs. 18.1–2. For higher degree curves, the neighborhood of the simple singularity looks similarly[1].

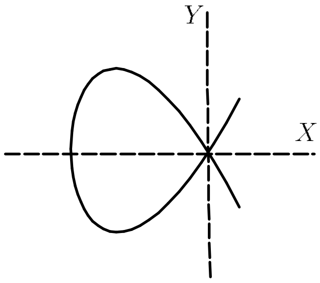

fig. 18.1. Node $y^2 = x^2(x+1)$.

Exercise 18.2. Show that irreducible cubic curve has at most one (automatically simple) singularity.

HINT. A line ℓ has to be a component of a cubic C as soon as $(\ell, C) \geqslant 4$.

AFFINE NEIGHBORHOOD OF A SINGULARITY. Let C have an equation $f(x,y) = 0$ in some affine chart with the origin at a singularity $p \in C$. We write the equation for C as $\sum f_\nu(x,y) = 0$, where each $f_\nu(x,y)$ is homogeneous of degree ν, and consider a line ℓ given parametrically as $x = \alpha t$, $y = \beta t$, where $q = (\alpha\colon \beta) \in \mathbb{P}_1$ is a point at the infinity. Then $C \cap \ell$ is given by

$$f_m(\alpha,\beta)\, t^m + f_{m+1}(\alpha,\beta)\, t^{m+1} + \cdots + f_d(\alpha,\beta)\, t^d = 0$$

and we see that $f_\nu(\alpha,\beta)$ is just the ν-th degree polar of p evaluated at $q = (\alpha\colon\beta)$. So, p is an m-typle point and the equation $\varphi_m(\alpha,\beta) = 0$ gives the directions of geometrical tangents through p (because $(C, \ell_{(\alpha:\beta)})_p$ jumps exactly along these directions).

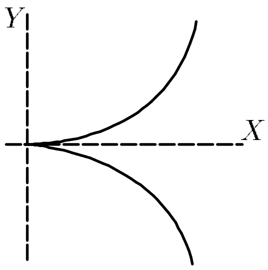

fig. 18.2. Cusp $y^2 = x^3$.

Example 18.3: analyzing singularities. Taking $x = \alpha t$, $y = \beta t$ in the nodal cubic equation $y^2 - x^2 - x^3 = 0$, we get the lowest term $(\beta+\alpha)(\beta-\alpha)\,t^2$, which vanishes for $(\alpha\colon\beta) = (1\colon \pm 1)$; so, there are two distinct geometrical tangents $x = \pm y$. Local intersection number $(\ell, C)_p = 3$ for each tangency, that is, each tangency has the second order contact with its branch. The cuspidal cubic on the fig. 18.2 has the lowest term $\beta^2\,t^2$. So, the second polar is a the double line $x = 0$ with local intersection 3 with C at the origin. As an advanced example, consider a quartic given by the polynomial $F(t) = t_0^4 - t_0^3 t_1 + t_0^2 t_2^2 - t_1^2 t_2^2$. Its singularities are $t \in \mathbb{P}_2$ where all partial derivatives

[1] one can show that any smooth curve in $\mathbb{P}_n$ admits a plane projection that has only the simplest singularities

$\dfrac{\partial F}{\partial t_0} = 4\,t_0^3 - 3\,t_0^2 t_2 - 2\,t_0 t_2^2\,,\ \dfrac{\partial F}{\partial t_1} = -2\,t_1 t_2^2\,,\ \dfrac{\partial F}{\partial t_2} = -t_0^3 + 2\,t_0^2 t_2 - 2\,t_1^2 t_2$. vanish simultaneously. It happens at two points $a = (0\colon 0\colon 1)$ and $b = (0\colon 1\colon 0)$. Take an affine chart with $x = t_0/t_2\,,\ y = t_1/t_2$ near a. Then $F = 0$ turns into $x^2 - y^2 - x^3 + x^4 = 0$ with two simple geometrical tangents $x = \pm y$ at the origin. Since a local intersection number equals 3 for each tangency, a is an ordinary node. Taking a chart with $x = t_0/t_1\,,\ y = t_2/t_1$ near b, we get the equation $y^2 - x^4 + x^3 y - x^2 y^2$ whose geometrical tangency is a double line $\ell = \{x = 0\}$ with $(\ell, C)_p = 4$. So, b is the non ordinary cusp, where C has a selfcontact of order 4.

<u>Blow up</u> . Geometrically, the substitution $x = \alpha t,\ y = \beta t$ lifts C from $\mathbb{P}_2$ to a surface $\mathcal{G} \subset \mathbb{P}_1 \times \mathbb{P}_2$ called *a blow up* of $p \in \mathbb{P}_2$ and described as follows. Identify a pencil of lines through p on $\mathbb{P}_2$ with the infinite line $\mathbb{P}_1 = (ab)$ (see fig. 18.3) and consider the incidence graph

$$\mathcal{G} \overset{\text{def}}{=} \{(\ell, q) \in \mathbb{P}_1 \times \mathbb{P}_2 \,|\, \ell \ni q\}\ .$$

Clearly, $\mathcal{G}$ is an algebraic surface in $\mathbb{P}_1 \times \mathbb{P}_2$, because, say, if $p = (1\colon 0\colon 0)$, $a = (0\colon 1\colon 0)$, $b = (0\colon 0\colon 1)$, then $(x_0\colon x_1\colon x_2) \in \mathbb{P}_2$ lies on (pq) with $q = \alpha a + \beta b$ iff $\alpha x_2 = \beta x_1$ (in affine chart with $x = x_1/x_0$, $y = x_2/x_0$ this turns into $\alpha y = \beta x$, what is equivalent to $x = \alpha t\,,\ y = \beta t$ for some t). The projection $\mathcal{G} \xrightarrow{\sigma} \mathbb{P}_2$ is bijective outside p, but $\sigma^{-1}(p) = \mathbb{P}_1$ stays in 1–1 correspondence with the lines through p on $\mathbb{P}_2$. A map

$$(\,(\alpha\colon \beta)\,,\, t) \longmapsto (\,(\alpha\colon \beta)\,,\,(1\colon \alpha t\colon \beta t)\,) \in \mathbb{P}_1 \times \mathbb{P}_2$$

gives a parametrization for $\mathcal{G}$ near this exceptional fiber. So, $\sigma^{-1}(C)$ consists of two components: $\mathbb{P}_1 = \sigma^{-1}(p)$ and a curve whose parametric equation is a result of the substitution $x = \alpha t$, $y = \beta t$ in the affine equation for C.

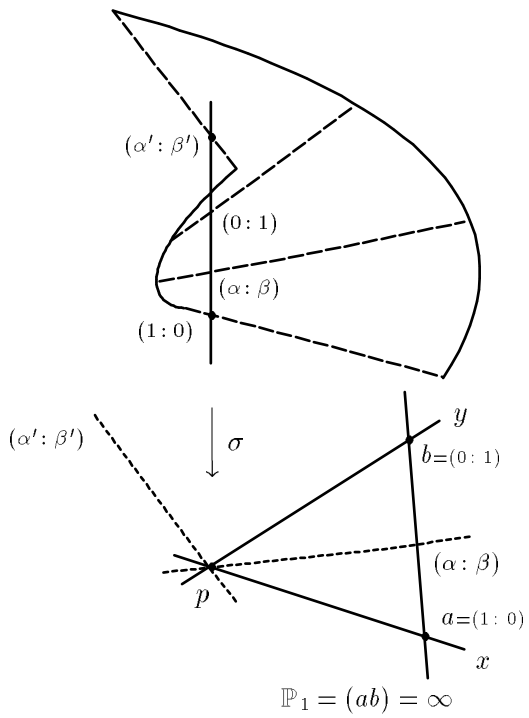

fig. 18.3. Blow up.

<u>Subspace of singular curves.</u> For a fixed $p \in \mathbb{P}_2$, a polar map $F \longmapsto \partial^{d-m} F / \partial p^{d-m}$ is a linear epimorphism $S^d V^* \longrightarrow S^m V^*$. So, the curves whose m-degree polar is a given collection of lines through p form a projective subspace of codimension $\dim(S^m V^*) - 1 = m(m+3)/2$. For example, we kill 5 parameters of a curve equation by assuming that this curve has a cusp with a given tangency at a given point. However, these restrictions, if come from several distinct points, are not independent, in general.

<u>Proper tangents and class.</u> A tangency (qp) through a smooth point $p \in C$ is called *proper*. Let $\deg C = d$. By Corollary 16.5, the tangents coming form $q \in \mathbb{P}_2$ meet C at $C \cap C_q^{(d-1)}$, where $C_q^{(d-1)}$ is $(d-1)$-th degree polar of q. If C is irreducible and q does not coincide with any singular point on C, then $C \cap C_q^{(d-1)}$ consists of $d(d-1)$ points[2] counted with multiplicities. Note that $C \cap C_q^{(d-1)}$ contains all singular points of C besides the points p such that (qp) is proper tangency at p. A number of proper tangents to C passing through a generic point $q \in \mathbb{P}_2$ is called *a class* of C and denoted by $c = c(C)$. Anyway, $c \leqslant d(d-1)$; the equality holds, if C is smooth.

<u>Inflections.</u> A smooth point $p \in C$ is called *an inflection* if $(C, T_pC)_p \geqslant 3$. An inflection is called *ordinary* or *simple* if this number equals 3. If $p \in C$ is an inflection, then the quadratic polar $C_p^{(2)}$ of p has the zero restriction onto $\ell = T_pC$, that is, ℓ is a component of $C_p^{(2)}$. Note that p is a smooth point of the conic $C_p^{(2)}$, because $C_p^{(2)}$ and C have the same linear polar of p and p is smooth on C. So, $p \in C$ is an inflection iff $C_p^{(2)} = \ell \cup \ell'$ with $\ell \cap \ell' \neq p$. The points $q \in \mathbb{P}_2$ with degenerate quadratic polar $C_q^{(2)}$ form a curve H_C, which is called *the Hessian* of C and is given by the equation $\det C_q^{(2)} = 0$ of degree $3(d-2)$ in q, where $d = \deg C$. Hence, an irreducible curve of degree $d \geqslant 3$ has at most $3d(d-2)$ inflections, which are contained in $C \cap H_C$. Note that this intersection contains also all singular points of C. ([3])

[2] If $\operatorname{char} k \neq 0$, this is not true, certainly. For example, all tangents to $x_0^2 = x_1 x_2$ pass through one point of $\mathbb{P}_2$, if $\operatorname{char} k = 2$.

[3] Of course, one can write a full system of precise relations between the class, the number of inflections, and the degree of a curve as soon as a number an and a type of singularities is fixed. These relations are called the *Plücker identities*. In the next lecture we will write them for the curves with simplest singularities.