Plane curves: duality and Plücker formulas.

(Warning: in this lecture we are working over $\mathbb{C}$ (or any algebraically closed field with char $=0$.)

DUAL CURVE Let $C \subset \mathbb{P}_2$ be an irreducible curve of degree d given by $F(x) = 0$. For any smooth $p \in C$ a tangency $\tau_p = T_pC$ defines a point $\tau_p^* = \operatorname{Ann} \tau_p \in \mathbb{P}^\times$ on the dual plane. When p varies along C, τ_p^* is running through an open dense set of some curve $C^\times \subset \mathbb{P}_2^\times$ called *a dual curve* for C.

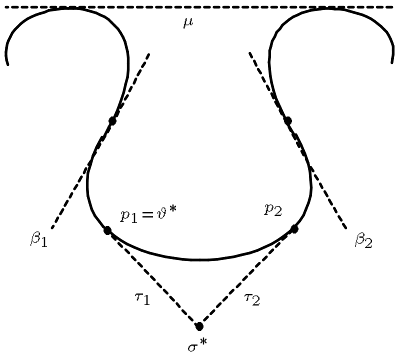

fig. 19.1. $C = C^{\times\times}$

Example 19.1: an equation for $C^\times$. Suppose that $(1:0:0) \notin C$ and let a line $\ell_\xi = \{\sum \xi_i x_i = 0\}$ have $\xi_0 \neq 0$. Then $C \cap \ell_\xi$ consists of all points $(-(\xi_1 x_1 + \xi_2 x_2) : \xi_0 x_1 : \xi_0 x_2) \in \mathbb{P}_2$ such that $F(-(\xi_1 x_1 + \xi_2 x_2), \xi_0 x_1, \xi_0 x_2) = 0$. This equation has a multiple root x iff its discriminant vanishes. The discriminant $D_F(\xi)$ is homogeneous polynomial in ξ of degree $d(d-1)$ and the equation $D_F(\xi) = 0$ describes all $\xi \in \mathbb{P}_2^\times$ such that $(C, \ell_\xi)_p \geqslant 2$ at some $p \in \mathbb{P}_2$. Clearly, this locus of ξ's split into a number of (multiple) lines representing the pencils centered at singular points of C and some nonlinear component representing the proper tangents[1]. Algebraically, this leads to factorization: $D_F(\xi) = F^*(\xi) \cdot \lambda_1(\xi)^{m_1} \cdot \cdots \cdot \lambda_s(\xi)^{m_s}$ where $\lambda_i(\xi)$ are linear. Nonlinear term F^* represents $C^\times$. Geometrically, the correspondence $C \overset{p \mapsto \tau_p^*}{\dashrightarrow} \mathbb{P}_2^\times$ is "a regular map" at the smooth points of C and "blows up" all the singularities of C by sending each of them to a (multiple) line.

Claim 19.2 $\quad C^{\times\times} = C$

Proof. A tangency $\vartheta \subset \mathbb{P}_2^\times$ at a smooth point $\tau_1^* \in C^\times$ is a limit of secant lines $\sigma = (\tau_1^*, \tau_2^*)$ when $\tau_2^* \longrightarrow \tau_1^*$ (see figs. 19.1–2). The secant σ represents a pencil of lines through $\sigma^* = \tau_1 \cap \tau_2 \in \mathbb{P}_2$. Clearly, $\sigma^* \longrightarrow p_1$ as $p_2 \longrightarrow p_1$. $\square$

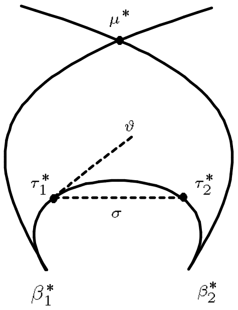

fig. 19.2. $C^\times$.

The figs. 19.1–2 show also that selfcontacts correspond to inflections and selfintersections correspond to *multiple tangents*[2] under the duality $C \longleftrightarrow C^\times$. In particular, if C has at most the simple singularities, then special proper tangents of $C^\times$ are exhausted by simple inflections and double tangents. Clearly, the class of C equals the degree of $C^\times$ and wise versa.

PLÜCKER IDENTITIES. Let C be a curve of class c and degree d. If its singularities are exhausted by δ ordinary selfintersections of multiplicities $m_1, m_2, \dots, m_n$ and $\varkappa$ ordinary cusps, then

$$c = d\,(d-1) - 3\,\varkappa - \sum_{\nu=1}^{\delta} m_\nu (m_{\nu-1}) \tag{31}$$

If we assume, in addition, that C has only ordinary inflections, then their number $\iota = \iota(C)$ equals

$$\iota = 3\,d\,(d-2) - 8\,\varkappa - 3 \sum_{\nu=1}^{\delta} m_\nu (m_{\nu-1}) \tag{32}$$

This formulas are known as *the Plücker identities*. In particular, if both C and $C^\times$ have at most the simple singularities, then the Plücker relations written for the both curves turn into

$$c = d\,(d-1) - 3\,\varkappa - 2\,\delta \qquad \iota = 3\,d\,(d-2) - 8\,\varkappa - 6\,\delta \qquad d = c\,(c-1) - 3\,\iota - 2\,\beta \qquad \varkappa = 3\,c\,(c-2) - 8\,\iota - 6\,\beta$$

where β is a number of bitangents for C. Any three of d, c, $\varkappa$, δ, β, ι can be found from these equations as soon as the other three are known.

Exercise 19.3. Let $q \notin C$ lie neither on an inflection tangency nor on a geometric tangency through a singular point of C; we write $C_q^{(d-1)}$ for $(d-1)$-th degree polar of q with respect to C. Deduce (31) from the equality $(C, C_q^{(d-1)}) = d(d-1)$ by proving that $(C, C_q^{(d-1)})_p$ equals 1 for smooth p, equals 3, if p is an ordinary cusp, and equals $m(m-1)$, if p is an ordinary m-typle selfintersection.

[1] compare this with Lemma 17.3

[2] a proper tangency is called *multiple* if it touch the curve in several distinct points

HINT. In the first case p is smooth on $C_q^{(d-1)}$ and $T_pC_q^{(d-1)} \neq T_pC$; in the second case p is smooth on $C_q^{(d-1)}$ again, but $T_pC_q^{(d-1)}$ coincides with the cuspidal tangency; in the third case p is an $(m-1)$-typle point on $C_q^{(d-1)}$, but each geometrical tangency of C at p is transversal to $C_q^{(d-1)}$, that is, intersects it with multiplicity $(m-1)$. Now, use the Zeuthen rule.

We'll prove (31), (32) using a geometrical approach traced back to Chasles, Cayley and Brill[3]

BLOW UP OF $\Delta \subset \mathbb{P}_2 \times \mathbb{P}_2$. Identify $\mathbb{P}_2^\times$ with the set of lines on $\mathbb{P}_2$ and consider the incidence graph

$$\mathcal{B} \stackrel{\text{def}}{=} \{(p,q,\ell) \mid p,q \in \ell\} \subset \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^\times$$

It is given by two quadratic equations $\sum \vartheta_\nu x_\nu = \sum \vartheta_\nu y_\nu = 0$, where $(x,y,\vartheta) \in \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^\times$. Topologically, $\mathcal{B}$ is a 4-dimensional compact manifold. A projection $\mathcal{B} \xrightarrow{\sigma} \mathbb{P}_2 \times \mathbb{P}_2$ is bijective outside the diagonal $\Delta \stackrel{\text{def}}{=} \{(p,p)\} \subset \mathbb{P}_2 \times \mathbb{P}_2$. Each fiber $\sigma^{-1}(p,p) = \{(p,p,\ell) \mid \ell \ni p\}$ over $(p,p) \in \Delta$ is naturally identified with the line pencil through p on $\mathbb{P}_2$; a 3-dimensional submanifold $E \stackrel{\text{def}}{=} \sigma^{-1}(\Delta) \subset \mathcal{B}$ is called *an exceptional divisor*. We write $A_1 = \pi_1^{-1}(\lambda)$, $A_2 = \pi_2^{-1}(\lambda)$, $M = \pi_3^{-1}(\lambda)$ for preimages of generic line λ under the projections of $\mathcal{B}$ onto plane factors in $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^\times$. Topologically, A_1, A_2, and M are 3-dimensional cycles on $\mathcal{B}$ and their homology classes don't depend on a choice of a line in each plane. Any 1-parametric algebraic family of "pointed" lines $(pq) \subset \mathbb{P}_2$ is pictured now by a curve $\Gamma \subset \mathcal{B}$ given by some algebraic equations. A topological location of such a curve is described by a triple of numbers:

$\alpha_1 = \#(\Gamma \cap A_1)$ — a number of p-points in $\Gamma = \{(p,q,\ell)\}$ laying on a generic line $\lambda \subset \mathbb{P}_2$;
$\alpha_2 = \#(\Gamma \cap A_2)$ — a number of q-points in $\Gamma = \{(p,q,\ell)\}$ laying on a generic line $\lambda \subset \mathbb{P}_2$;
$\mu = \#(\Gamma \cap M)$ — a number of lines ℓ in $\Gamma = \{(p,q,\ell)\}$ passing through a generic point $c \in \mathbb{P}_2$ ([4]).

Strongly speaking, we should use the topological intersection indices instead of "the numbers of points". But for all Γ below there will be an open dense set of lines[5] on each plane factor in $\mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^\times$ such that corresponding A_1, A_2, M will intersect Γ transversally in a constant finite number of points; so, we will always suppose α_1, α_2, μ calculated by use A_1, A_2, M coming from these open dense sets[6]. The triples $(p,q,\ell) \in \Gamma$ having $p = q$, that is, the points of $\Gamma \cap E$, are called *exceptional*. Typically, Γ has a finite number of exceptional points. Our goal is to equip the exceptional points with appropriate multiplicities and express the number $\xi(\Gamma)$ of "exceptional points counted with multiplicities" through α_1, α_2, μ in the following three examples.

Example 19.4: a join family. Let $C_1, C_2 \subset \mathbb{P}_2$ be two curves of degrees d_1, d_2 without common components and $c \in \mathbb{P}_2$ be a point outside them and outside the lines joining distinct intersections of C_1, C_2 as well. Then

$$\Gamma = \{(p,q,\ell) \in \mathcal{B} \mid p \in C_1,\ q \in C_2,\ \ell \ni c\}$$

is a curve in $\mathcal{B}$ given by an obvious triple of algebraic equations. Its exceptional points are $(p,p,(pc))$ with $p \in C_1 \cap C_2$, that is, $\xi(\Gamma)$ counts the intersections of C_1 and C_2. Further, $\alpha_1 = \alpha_2 = d_1 d_2$, because a generic line intersect, say, C_1 in distinct points $p_1, p_2, \dots, p_{d_1}$ and there are d_2 intersections of C_2 with (cp_ν) for each ν. Finally, $\mu = d_1 d_2$ too, because there is a unique line $(c\lambda^*)$ passing through a given λ^* and this line contains $d_1 d_2$ distinct pairs (p,q) with $p \in C_1$, $q \in C_2$, if O is general enough.

Example 19.5: a secant family. In the above example, let $C_1 = C_2 = C$ be the same irreducible curve of degree d given by an equation $F(x) = 0$ and c be a point outside singular tangents and lines joining distinct singularities. Then a closure of $\{(p,q,\ell) \in \mathcal{B} \mid p \neq q,\ p,q \in C,\ \ell \ni c\}$ is a curve given by the equations $F(x) = F(y) = \vartheta(c) = 0$ in $(x,y,\vartheta) \in \mathbb{P}_2 \times \mathbb{P}_2 \times \mathbb{P}_2^x$. Exceptional points of this curve are $(p,p,(pc))$ with either $(pc) = T_pC$ or $p \in Sing(C)$. So, ξ counts singular points of C and proper tangents coming from c to C. By the same reasons as above, $\alpha_1 = \alpha_2 = \mu = d\,(d-1)$ in this case.

Example 19.6: a tangent family. Let C be as above and Γ be a closure of $\{(p,q,\ell) \in \mathcal{B} \mid p \neq q,\ p,q \in C,\ \ell = T_pC\}$ that is, Γ is given by the equations $F(x) = F(y) = \dfrac{\partial F}{\partial y}(x) = 0$. Exceptional points (p,p,ℓ) of Γ are either smooth p with the inflection tangency $\ell = T_p$ or singular p with a geometrical tangency ℓ at p. So, $\xi(\Gamma)$ counts inflections and singular tangents. Since a simple tangent at a smooth point p intersect the curve in $(d-2)$ more points q, we have $\alpha_1 = d\,(d-2)$. Clearly, $\mu = c\,(d-2)$, because there are c proper tangents to C through generic $c \in \mathbb{P}_2$ by the definition of class. Finally, $\alpha_2 = d\,(c-2)$, because there are $(c-2)$ tangents to C coming from a generic $q \in C$. To see this, we need a more precise analysis of the previous example. It will be done in the next lecture.

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[3] We follow the book: J. G. Semple, L. Roth, *Introduction to algebraic geometry*, Oxford, 1949.
[4] here $\lambda = c^* \subset \mathbb{P}_2^\times$ is a pencil of lines through c
[5] in the space of all lines, that is, on the dual plane
[6] it will be an exercise for readers, to check the existence of such open sets of lines in each of examples below