Projective spaces: coordinates, affine coverings, and subspaces.

<u>GLOBAL HOMOGENEOUS COORDINATES.</u> If a basis $\{e_0, e_1, \dots, e_n\} \subset V$ is fixed, then any vector $v \in V$ can be identified by its coordinate row: we write $v = (x_0, x_1, \dots, x_n)$ for $v = x_0 e_0 + x_1 e_1 + \dots + x_n e_n$. Two vectors $v = (x_0, x_1, \dots, x_n)$ and $w = (y_0, y_1, \dots, y_n)$ represent the same point $p \in \mathbb{P}_n$ iff they are proportional, i. e. $x_\nu = \lambda\, y_\nu\ \forall \nu$ for some $\lambda \neq 0$. Hence, only the ratios $x_\mu : x_\nu$ are well defined for the point $p \in \mathbb{P}_n$. These ratios are called *homogeneous coordinates* on $\mathbb{P}_n$. We write $p = (x_0 \colon x_1 \colon \ \cdots : x_n)$ for the point p that is represented by $v = (x_0, x_1, \dots, x_n)$ up to proportionality. Any homogeneous polynomial equation on the homogeneous coordinates produces the projective algebraic subset in $\mathbb{P}_n$. For example, the equation

$$x_0^2 + x_1^2 = x_2^2 \tag{2}$$

presents the curve $C \subset \mathbb{P}_2$ known as *non degenerate plane conic* (if $\operatorname{char} k \neq 2$).

<u>LOCAL AFFINE COORDINATES.</u> Linear forms on V induce also a lot of local affine coordinate systems. Each of them is defined only inside a given affine chart U but gives a one-to-one presentation for the points of U. Let $U \subset \mathbb{A}(V)$ be given by the equation $\varphi(x) = 1$, where $\varphi \in V^*$. Then, for example, the restrictions of coordinate forms $t_i \stackrel{\text{def}}{=} x_i|_U$ are affine linear functions on U and appropriate n of them[1] can be taken as coordinates inside U. In order to calculate the values of these affine coordinates for a given point $p = (p_0 \colon p_1 \colon \ \cdots : p_n) \in \mathbb{P}_n$, we just have to represent this point by the vector $u \in U$. Certainly,

$$u = \left(\frac{p_0}{\alpha(p)}, \frac{p_1}{\alpha(p)}, \dots, \frac{p_0}{\alpha(p)}\right)$$

because of $\alpha(u) = \alpha(p/\alpha(p)) = \alpha(p)/\alpha(p) = 1$. Hence, our local coordinates are expressed by the homogeneous ones as $t_i = x_i/a(x)$. Note, that the sentences: "$\alpha(p) = 0$", "$t_i = \infty$ for some i", and "$p \in U_\infty = \mathbb{P}_n \setminus U$" really mean the same — p has run away from U to infinity.

<u>Example 2.1:</u> affine conics. Consider the non degenerate conic (2) over $k = \mathbb{R}$ in some affine charts. Let U_0 be given by the equation $\{x_0 = 1\}$. Then $t_1 = x_1|_{U_0} = x_1/x_0$ and $t_2 = x_2|_{U_0} = x_2/x_0$ can be taken as the local affine coordinates. Dividing the both sides of (2) by x_2^2, we get for $C \cap U_0$ the equation $t_2^2 - t_1^2 = 1$, i. e. it is the hyperbola. Similarly, for the chart $U_2 = \{x_0 = 1\}$ with the local affine coordinates $t_0 = x_0|_{U_2} = x_0/x_2$ and $t_1 = x_1|_{U_2} = x_1/x_2$ we get the equation $t_0^2 + t_1^2 = 1$, i. e. $C \cap U_2$ is the circle. Finally, consider the chart U given by $x_2 - x_1 = 1$ and take

$$t = x_0|_U = \frac{x_0}{x_2 - x_1} \qquad \text{and} \qquad u = (x_2 + x_1)|_U = \frac{x_2 + x_1}{x_2 - x_1}$$

as the local affine coordinates. Dividing by $(x_2 - x_1)^2$, we get after some elimination that $C \cap U$ is the parabola $u = t^2$.

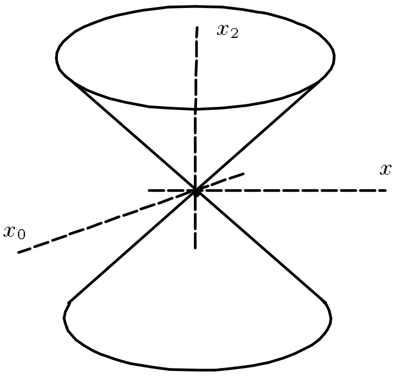

fig. 2.1. The cone.

Exercise 2.2. The affine cone $x_0^2 + x_1^2 = x_2^2$ in $\mathbb{A}^3$ is drawn on the fig. 2.1. Picture there each of 3 previous affine charts and outline their intersections with the cone.

HINT. It's good idea, to draw 3 separated pictures!

<u>PROJECTIVE CLOSURE.</u> Any closed affine algebraic set $X \subset \mathbb{A}^n$ is just the affine piece of appropriate closed projective algebraic set $\widetilde{X} \subset \mathbb{P}_n$, which is called *a projective closure of* X. If X is given by the polynomial equations $\{\, f_\nu(t_1, t_2, \dots, t_n) = 0 \,\}$, then we substitute $t_i = x_i/x_0$ and multiply the ν-th equation by $x_0^{\deg f_\nu}$. The resulting equations $\widetilde{f}_\nu(x_0, x_1, \dots, x_n) = 0$ are homogeneous and define the closed projective algebraic set $\widetilde{X} \subset \mathbb{P}_n$ such that $\widetilde{X} \cap U_0 = X$, where U_0 is the affine chart given by $x_0 = 1$ and is identified with original $\mathbb{A}_n$ by taking $t_i = x_i|_{U_0}$.

<u>STANDARD AFFINE COVERING AND GLUING RULES.</u> The $(n+1)$ affine charts $U_\nu \subset \mathbb{A}^{n+1}$ given by $\{x_\nu = 1\}$ cover together the whole projective space $\mathbb{P}_n$. This covering is called *standard*. For a given ν, $0 \leqslant \nu \leqslant n$, the restrictions

$$t_i^{(\nu)} = x_i|_{U_\nu} = \frac{x_i}{x_\nu}\,, \quad i \neq \nu$$

[1] these n should have linearly independent linear homogeneous parts

(where i runs only trough n of the values $0, 1, \ldots, n$ except for ν) are taken as *a standard local affine coordinates* on U_ν.

Topologically, $\mathbb{P}_n$ is constructed from $(n+1)$ distinct copies $(U_0, U_1, \ldots, U_n)$ of $\mathbb{A}_n$ by gluing along the actual intersections $U_\mu \cap U_\nu \subset \mathbb{P}_n$: we have to identify a point of U_ν with a point of U_μ, if they correspond to the same point of $\mathbb{P}_n$. In terms of the homogeneous coordinates, the intersection $U_\mu \cap U_\nu$ consists of all x such that both x_μ and x_ν are non zero. In affine coordinates inside U_μ, this intersection is given by the inequality $t_\nu^{(\mu)} \neq 0$. Inside U_ν it is given by $t_\mu^{(\nu)} \neq 0$. So, a point $t^{(\mu)} \in U_\mu$ is glued with a point $t^{(\nu)} \in U_\nu$ iff

$$t_\nu^{(\mu)} = 1/t_\mu^{(\nu)} \qquad \text{and} \qquad t_i^{(\mu)} = t_i^{(\nu)}/t_\mu^{(\nu)} \quad \text{for} \quad i \neq \mu, \nu$$

Right hand sides of these equations are called *transition functions* from $t^{(\nu)}$ to $t^{(\mu)}$ over $U_\mu \cap U_\nu$. For example, $\mathbb{P}_1$ can be produced from two copies of $\mathbb{A}^1$ by identifying the point t in one of them with the point $1/t$ in the other for all $t \neq 0$.

Exercise 2.3*. If you have some experience in smooth topology, prove that over $\mathbb{R}$ (corr. over $\mathbb{C}$) the projective line $\mathbb{P}_1$ is the smooth[2] manifold isomorphic to the circle S^1 (corr. to the Riemann sphere S^2).

<u>PROJECTIVE SUBSPACES.</u> A closed projective algebraic subset $L \subset \mathbb{P}_n = \mathbb{P}(V)$ given by a system of *linear* homogeneous equations is called *a projective subspace* in $\mathbb{P}_n$. It always has the form $L = \mathbb{P}(W)$, where $W \subset V$ is a vector subspace. Note that 0-dimensional projective subspaces[3] are exactly the points. Since $\operatorname{codim}_{\mathbb{P}(V)}\mathbb{P}(W) = \operatorname{codim}_V W$, we have $L_1 \cap L_2 \neq \varnothing$ for any two projective subspaces L_1 and L_2 such that $\operatorname{codim} L_1 + \operatorname{codim} L_2 \leqslant n$. For example, any two lines on $\mathbb{P}_2$ have to cross each other (in terms of $\mathbb{A}^3$: any two planes, which pass trough the origin, have a common line).

Two projective subspaces L_1 and L_2 in $\mathbb{P}_n$ are called *complementary* to each other, if

$$L_1 \cap L_2 = \varnothing \qquad \text{and} \qquad \dim L_1 + \dim L_2 = n - 1\,.$$

For example, any two skew lines in 3-dimensional space are complementary.

Exercise 2.4. Show that $\mathbb{P}(U)$ and $\mathbb{P}(W)$ are complementary in $\mathbb{P}(V)$ iff $V = U \oplus W$.

<u>Lemma 2.5</u> *If $L_1 = \mathbb{P}(U)$ and $L_2 = \mathbb{P}(W)$ are complementary in $\mathbb{P}_n = \mathbb{P}(V)$, then for any point $p \in \mathbb{P}_n \setminus (L_1 \cup L_2)$ there exists a unique line ℓ that pass through p and intersect both L_1 and L_2.*

Proof. Since $V = U \oplus W$, each $v \in V$ is uniquely represented in the form $v = u + w$, where $u \in U$, $w \in W$. The both components are non zero iff $v \notin U \cup W$. In this case there exists only one 2-dimensional subspace $T \subset V$ such that $v \in T$, $T \cap U \neq 0$, and $T \cap W \neq 0$. It coincides with the linear span of u and w. The line ℓ is just $\mathbb{P}(T)$. $\square$

[2] moreover, analytic
[3] i. e. $\mathbb{P}(k^1)$