Plane curves: Chasles - Cayley - Brill formula.

(Warning: in this lecture we are working over $\mathbb{C}$ (or any algebraically closed field with $\mathrm{char} = 0$.)

A CORRESPONDENCE ON $\mathbb{P}_1$ is called *algebraic of type m-n* if the pairs of corresponding points $(p,q) \in \mathbb{P}_1 \times \mathbb{P}_1$ form an algebraic curve $\gamma \subset \mathbb{P}_1 \times \mathbb{P}_1$ given by an irreducible bihomogeneous polynomial $g(t',t'')$ of bidegree (m,n) in $(t',t'') = \big((t'_0 : t'_1),\, (t''_0 : t''_1) \big)$. This curve is called *a graph* of the correspondence. So, *images* of a given point $p \in \mathbb{P}_1$ are presented by the equation $g(p,t) = 0$ in $t \in \mathbb{P}_1$ and *preimages* — by the equation $g(t,p) = 0$. Since these equations have respectively m and n ordinary distinct roots for almost all[1] p, almost all points have n distinct images and m distinct preimages under an m-n correspondence. A point $e \in \mathbb{P}_1$ is called *fixed* if it corresponds to itself[2], that is, if $g(e,e) = 0$. So, the fixed point set coincides with $\gamma \cap \Delta$, where $\Delta = \{(t,t)\} \subset \mathbb{P}_1 \times \mathbb{P}_1$ is the diagonal. Since $g(t,t)$ is homogeneous of degree $m+n$ in $t = (t_0, t_1)$, any algebraic m-n correspondence has $m+n$ fixed points counted with multiplicities, where the multiplicity of a fixed point e is nothing more than the standard local intersection number $(\gamma, \Delta)_{(e,e)}$ of the curve $g(t',t'') = 0$ with the line $t' = t''$ at $(e,e) \in \gamma$ and can be calculated by the Zeuthen rule in any affine chart $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}_2 \ni (e,e)$ as it was explained in the Lect. 17.

Claim 20.1 *Let $U \subset \mathbb{P}_1$ be an affine chart with the origin at a fixed point e of a correspondence γ, x be an affine coordinate on U, and $y_1(x), y_2(x), \dots, y_m(x)$ be all γ-images of x which tend to 0 as $x \longrightarrow 0$. Then $(\gamma, \Delta)_{(0,0)}$ equals the sum of orders of infinitesimals $y_i(x) - x$ with respect to x.*

Proof. Let (x,y) be affine coordinates on $\mathbb{A}^2 = U \times U$ (see fig. 20.1). Since the both lines $x = 0$ and $y = 0$ contain just one intersection point $(0,0) \in \gamma \cap \Delta$ we can use the line pencil $x = \mathrm{const}$ parametrized by the x-axis to calculate $(\gamma, \Delta)_{(0,0)}$ as the sum of orders of infinitesimals $\alpha_i(x) - \beta_j(x)$ where α_i and β_j run trough the intersections of a vertical line $x = \mathrm{const}$ respectively with γ and with Δ. So, $\alpha_i(x) = y_i(x)$ and there is just one $\beta(x) = x$. $\square$

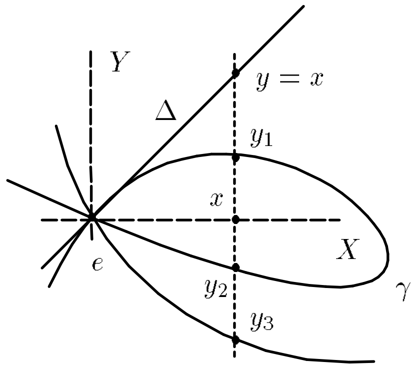

fig. 20.1. Fixed point

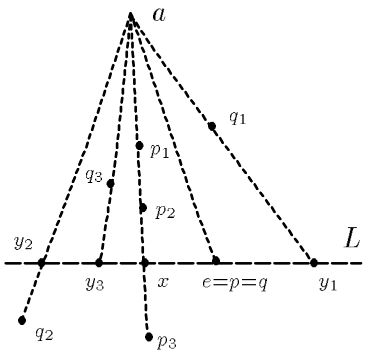

fig. 20.2. Exceptional point

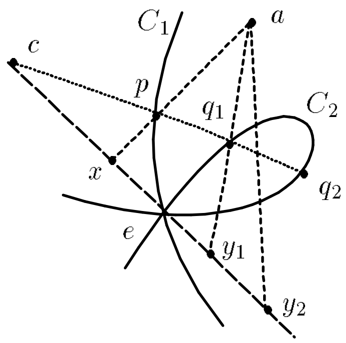

fig. 20.3. Intersection point

EXCEPTIONAL POINT MULTIPLICITIES. Let $\Gamma \subset \mathcal{B}$ be a family of pointed lines (p,q,ℓ) from the last lecture. Fix a point $a \in \mathbb{P}_2$ such that: 1) a lies exactly on μ distinct lines ℓ of Γ; 2) a generic line through a contains exactly α_1 distinct p-points and exactly α_2 distinct q-points of Γ. Then Γ induces an algebraic α_1-α_2 correspondence γ_Γ on the pencil of lines passing through a by the rule: $(ap) \longleftrightarrow (aq)$ if $(p,q,\ell) \in \Gamma$ for some[3] ℓ. A line through a corresponds to itself under γ_Γ either if it belongs to Γ or if it joins a with an exceptional point e such that $(e,e,\ell) \in \Gamma$ for some ℓ. We define *a multiplicity* of an exceptional point $(e,e,\ell) \in \Gamma$ as a multiplicity of the corresponding fixed point (ae) of the correspondence γ_Γ, that is, as $(\gamma_\Gamma, \Delta)_{((ae),(ae))}$, where $\Delta \subset \mathbb{P}_1 \times \mathbb{P}_1$ is the diagonal. It can be calculated geometrically by the Zeuthen rule as follows. Parametrize the pencil of lines through a by some line $L \not\ni a$ passing through an exceptional point e (see fig 20.2) and take an affine coordinate x on L near e. Let $(ay_1), (ay_2), \dots, (ay_m)$ be all lines corresponding to (ax) and tending to (ae) as $x \longrightarrow e$. Then the multiplicity of (pe) equals the sum of orders of infinitesimals $y_i(x) - x$ with respect to $x - e$.

Claim 20.2 *A total number of exceptional points counted with multiplicities equals*[4] $\xi(\Gamma) = \alpha_1 + \alpha_2 - \mu$.

Proof. Since $\deg(\gamma_\Gamma) = (\alpha_1, \alpha_2)$, it has $\alpha_1 + \alpha_2$ fixed points. Each line through a from Γ has multiplicity 1 as a fixed point for γ_Γ, because of the choice of a. The sum of other fixed point multiplicities equals $\xi(\Gamma)$. $\square$

[1] that is, for all p outside some finite subset on $\mathbb{P}_1$ where the both discriminants vanish

[2] of course, a fixed point may have also several other images and preimages besides itself

[3] of course, $\ell = (pq)$, if $p \neq q$

[4] this formula is known as *the Chasles - Cayley - Brill formula*

Now, let us return to three examples from the last lecture.

Example 20.3: exceptional point multiplicities in Example 19.4. Clearly, in this case the multiplicity of an exceptional point $(e, e, (ce)) \in \Gamma$ coincides with the local intersection number $(C_1, C_2)_e$ defined in the Lect. 19 (see fig. 20.3). So, $\xi(\Gamma)$ is the sum of local intersection numbers of C_1 and C_2. The Chasles - Cayley - Brill formula

$$\xi(\Gamma) = \alpha_1 + \alpha_2 - \mu = d_1 d_2 + d_1 d_2 - d_1 d_2 = d_1 d_2$$

leads to the Bézout theorem in this case.

Example 20.4: exceptional point multiplicities in Example 19.5. In the case $C_1 = C_2 = C$ there are 3 types of exceptional points. A proper tangency $(e, e, (ce))$ (see fig. 20.4) has multiplicity 1 because any x closed to e has a unique image y coming to e when $x \longrightarrow e$ and $y - x$ and $x - e$ are like infinitesimals. If e is an ordinary m-typle selfintersection, then a line through c closed to e contains m p-points cut out by m branches of C; each p-point produces $(m-1)$ q-points coming from remaining $(m-1)$ branches; so, there are $m(m-1)$ differences $y - x$ and each of them is like $(x - e)$ (see fig. 20.5). Finally, there are 2 branches of C through an ordinary cusp (see fig. 20.6). If we parametrize a line pencil through a by the cuspidal tangency, then each branch produces a unique difference $(y - x)$ of order $(x - e)^{3/2}$; so, cuspidal point has multiplicity 3. Hence,

$$c + \sum_{\nu}^{\delta} m_\nu (m_\nu - 1) + 3\,\varkappa = \xi(\Gamma) = \alpha_1 + \alpha_2 - \mu = d(d-1) + d(d-1) - d(d-1) = d(d-1)$$

by the Chasles - Cayley - Brill formula. This proves the first Plücker identity.

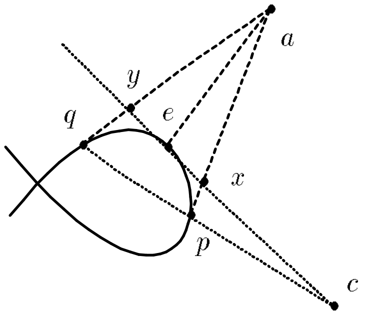

fig. 20.4.

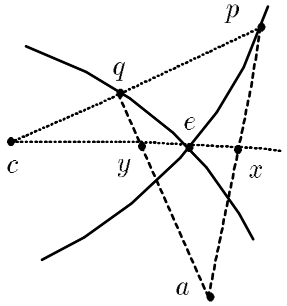

fig. 20.5.

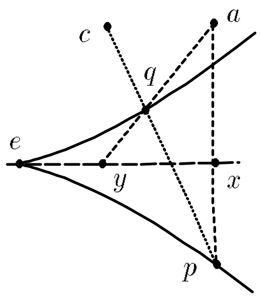

fig. 20.6.

Example 20.5: exceptional point multiplicities in Example 19.6. The figs. 20.7–9 show that each difference $(y - x)$ is like $(x - e)$ for all three types of exceptional points. An inflection point (fig. 20.7) produces a unique difference and has a multiplicity 1; each of m branches through an m-typle selfintersection (fig. 20.8) produces $(m-1)$ differences, so the multiplicity equals $m(m-1)$ here; a cusp (fig. 20.9) produces 2 differences and has multiplicity 2. Hence,

$$\iota + \sum_{\nu}^{\delta} m_\nu (m_\nu - 1) + 2\,\varkappa = \xi(\Gamma) = \alpha_1 + \alpha_2 - \mu = d(d-2) + d(c-2) - c(d-2) = d^2 - 4\,d + 2\,c$$

by the Chasles - Cayley - Brill formula. Since $c = d(d-1) - \sum m_\nu(m_\nu - 1) - 3\,\varkappa$, we get the second Plücker identity $\iota = 3\,d(d-2) - 3\sum m_\nu(m_\nu - 1) - 8\,\varkappa$.

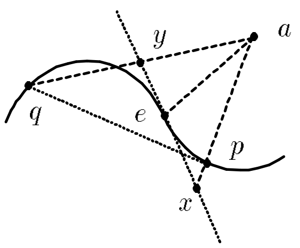

fig. 20.7.

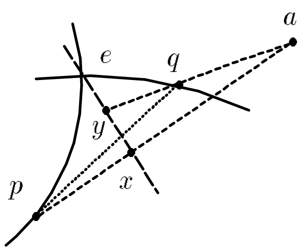

fig. 20.8.

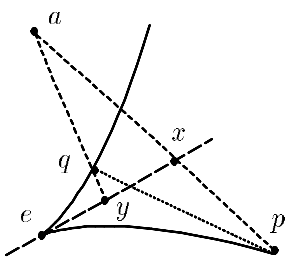

fig. 20.9.

Exercise 20.6. Let c be a smooth point of C in Example 19.5 and Example 20.4. Check that a multiplicity of an exceptional point $(c, c, T_c C) \in \Gamma$ equals 2 (so, we have really $\alpha_2 = d(c-2)$ in Example 19.6).