Algebraic – geometric dictionary.

AFFINE ALGEBRAIC SETS AND THEIR IDEALS. Let $X \subset \mathbb{A}^n$ be an affine algebraic set given by a polynomial equation system $f_\nu(x_1, x_2, \dots, x_n) = 0$. Clearly, $X = V(I)$, where $I \subset k[x_1, \dots, x_n]$ is an ideal generated by f_ν. Algebraically, I is obtained from the initial system by adding all its consequences $\sum g_\nu f_\nu = 0$ with $g_\nu \in k[x_1, \dots, x_n]$. Of course, $V(I) \subset V(J) \iff J \subset I$. Restrictions of the polynomials $f \in k[x_1, \dots, x_n]$ onto X produce k-valued functions $X \xrightarrow{v \mapsto f(v)} k$ called *regular algebraic functions* on X. They form a finitely generated commutative k-algebra $k[X]$ called *a coordinate algebra* of X. Clearly, $k[X] = k[x_1, \dots, x_n]/I(X)$, where $I(X)$ is an ideal of all polynomials vanishing along X. By Hilbert's Nullstellensatz, if $X = V(I)$ and k is algebraically closed, then $I(X) = \{ f \in k[x_1, \dots, x_n] \mid f^n \in I \text{ for some } n \in \mathbb{N} \}$ is *a radical* of the initial equation ideal I defining X.

Example 22.1: one point sets. An affine algebraic set $V(I) \subset \mathbb{A}(V)$ coincides with the origin $\{O\} \in \mathbb{A}(V)$ iff $I \supset S^d V^*$ for all $d \gg 0$. Indeed, for each i some power $x_i^{m_i} \in I$, because each x_i vanishes at O. So, $S^d V^* \subset I$ for $d \geqslant 1 + \sum (m_\nu - 1)$. Conversely, $V(I) = \{O\}$ if I contains some powers $x_i^{m_i}$ of all coordinate functions.

Exercise 22.2. Prove that $I(\{O\})$ is a maximal ideal and is generated by x_i.

Example 22.3: resultant systems. We fix a collection of $(n+1)$ degrees $d_0, d_1, \dots, d_n$ and denote by $\mathcal{S}_d = \mathbb{P}(S^d V^*)$ the space of hypersurfaces of degree d in $\mathbb{P}_n = \mathbb{P}(V)$. Let $\mathcal{R} \subset \mathcal{S}_{d_0} \times \mathcal{S}_{d_1} \times \dots \times \mathcal{S}_{d_n}$ be a set of all hypersurface collections $S_0, S_1, \dots, S_n \subset \mathbb{P}_n$ such that $\bigcap S_\nu \neq \varnothing$. We are going to prove that $\mathcal{R}$ is a projective algebraic set, that is, is given by a finite system of multihomogeneous polynomial equations in the coefficients of forms $(f_0, f_1, \dots, f_n) \in S^{d_0} V^* \times S^{d_1} V^* \times \dots \times S^{d_n} V^*$ presenting the hypersurfaces $S_0, S_1, \dots, S_n$ (this equation system depends only on n, $d_0, d_1, \dots, d_n$ and is called *a resultant system*). Let $I \subset k[x_0, x_1, \dots, x_n]$ be an ideal generated by f_ν; then $\bigcap S_\nu \subset \mathbb{P}(V)$ is empty $\iff V(I) \subset \mathbb{A}(V)$ coincides with the origin. So, $\bigcap S_\nu \neq \varnothing \iff S^d V^* \subsetneq I \ \forall d \iff$ a k-linear map $\mu_d : S^{d-d_0} V^* \oplus S^{d-d_1} V^* \oplus \dots \oplus S^{d-d_n} V^* \xrightarrow{(g_0, g_1, \dots, g_n) \mapsto \sum g_\nu f_\nu} S^d$ is not surjective $\forall d$. In terms of the standard monomial bases, μ_d is presented by a matrix whose entries are linear forms in the coefficients of f_ν. So, $\mathcal{R}$ is given by vanishing of all maximal minors of all matrices μ_d for d such that $\sum\limits_{\nu=0}^{n} \binom{n+d-d_\nu}{n} \geqslant \binom{n+d}{n}$ (this takes place for all d large enough and means that the left side dimension is greater than the right side one). By Hilbert's base theorem, this system of homogeneous equations is equivalent to some finite subsystem. For example, we have seen in Lect. 17 that if $n = 1$, then two polynomials of degrees d_0, d_1 have a common zero on $\mathbb{P}_1$ iff their resultant vanishes. So, in this case $\mathcal{R}$ is a hypersurface; its equation coincides with $\det \mu_d = 0$ for $d = d_0 + d_1$ — this is the first value of d producing a non trivial restriction: the dimensions of the both sides coincide.

Example 22.4: a projection $\mathbb{P}_m \times \mathbb{A}^n \xrightarrow{\pi} \mathbb{A}^n$ sends algebraic sets to algebraic sets, that is, any set $\{(t, x) \in \mathbb{P}_m \times \mathbb{A}^n\}$ given by polynomial equations $f_\nu(t, x) = 0$ homogeneous in $t = (t_1, t_2, \dots, t_m) \in \mathbb{P}_m$ is projected onto some affine algebraic subset of $\mathbb{A}^n$. Indeed, consider f_ν as homogeneous polynomials in t with the coefficients in $k[x_1, \dots, x_n]$. Then the projection on $\mathbb{A}^n$ consists of all p such that $f_\nu(t, p)$ have a common zero on $\mathbb{P}_m$. By the Example 22.3, this means that their coefficients satisfy the resultant equation system. So, such p form an algebraic set.

Claim 22.5 *Let A be a finitely generated algebra over an algebraically closed field. Then $A = k[X]$ for some affine algebraic set X iff A doesn't contain nilpotents (that is, $a^n = 0 \Rightarrow a = 0$; such an algebra is called reduced).*

Proof. An algebra of functions with values in a field is obviously reduced. Conversely, if a finitely generated k-algebra $A = k[x_1, \dots, x_n]/I$ is reduced, then $f^n \in I \Rightarrow f \in I$ for any $f \in k[x_1, \dots, x_n]$. By Hilbert, this means that $X = V(I) \subset \mathbb{A}^n$ has $I(X) = I$, if k is algebraically closed. So, $A = k[x_1, \dots, x_n]/I(X) = k[X]$. □

POINTS. All regular functions $f \in k[X]$ vanishing at a given point $p \in X$ form a maximal proper ideal $\mathfrak{m}_p \subset k[X]$, because of $k[X]/\mathfrak{m}_p = k$. This ideal is called *a maximal ideal* of p. Note that an evaluation map $ev_p : k[X] \xrightarrow{f \mapsto f(p)} k$ coincides with the factorization epimorphism $k[X] \xrightarrow{f \mapsto f \,(\mathrm{mod}\, \mathfrak{m}_p)} k[X]/\mathfrak{m}_p = k$.

Claim 22.6 *If k is algebraically closed, then the correspondences $p \longleftrightarrow \mathrm{ev}_p \longleftrightarrow \mathfrak{m}_p = \ker(\mathrm{ev}_p)$ establish bijections between the points of X, the homomorphisms $k[X] \longrightarrow k$ identical on k, and the proper maximal ideals in $k[X]$.*

Proof. The second correspondence is obviously bijective. Clearly, $p \neq q \Rightarrow \mathfrak{m}_p \neq \mathfrak{m}_q$, because there is, say, a linear form $\mathbb{A}^n \xrightarrow{\ell} k$ such that $\ell \in \mathfrak{m}_p$ but $\ell \notin \mathfrak{m}_q$. Finally, consider a proper maximal ideal $\mathfrak{m} \subset k[X]$ and write $k[X]$ as $k[x_1, \dots, x_n]/I(X)$. A full preimage $\widetilde{\mathfrak{m}} \subset k[x_1, \dots, x_n]$ of $\mathfrak{m}$ is a proper maximal ideal, because of $\widetilde{\mathfrak{m}} \supset I(X)$ and $k[x_1, \dots, x_n]/\widetilde{\mathfrak{m}} = k[X]/\mathfrak{m} = k$. Hence, $V(\widetilde{\mathfrak{m}}) \subset \mathbb{A}^n$ is nonempty and is contained in X, that is, there is a point $p \in X$ such that $f(p) = 0$ for any $f \in \mathfrak{m}$. So, $\mathfrak{m} \subset \mathfrak{m}_p$ and $\mathfrak{m} = \mathfrak{m}_p$, because $\mathfrak{m}$ is maximal. □

ALGEBRAIC SETS VIA SPECTRA. A set of all proper maximal ideals in a given k-algebra A is called *a maximal spectrum* of A and is denoted by $Spec_m(A)$. In pure algebraic (and coordinateless) terms, an affine algebraic set over an algebraically closed field k is a maximal spectrum $Spec_m(A)$ of a finitely generated reduced k-algebra A whose elements $f\in A$ are considered as the functions $Spec_m(A) \xrightarrow{\mathfrak{m}\mapsto f\,(\mathrm{mod}\,\mathfrak{m})} k$.

<u>Regular morphisms of algebraic sets.</u> Any map of sets $X \xrightarrow{\varphi} Y$ produces *a pull back homomorphism* φ^* from the algebra off all k-valued functions on Y to the one on X; it sends $Y \xrightarrow{f} k$ to the composition $\varphi^* f : X \xrightarrow{\varphi} Y \xrightarrow{f} k$. A map $X \xrightarrow{\varphi} Y$ between affine algebraic sets is called *a regular morphism* of algebraic sets if φ^* sends the regular algebraic functions to the regular algebraic functions on X, that is, induces a well defined homomorphism of coordinate rings $k[Y] \xrightarrow{\varphi^*} k[X]$.

<u>Claim 22.7</u> *Let $B \xrightarrow{\varphi^*} A$ be a homomorphism of finitely generated reduced algebras over algebraically closed field k such that $\varphi^*(1) = 1$. Then φ^* is a pull back homomorphism for a unique regular map $Spec_m(A) \xrightarrow{\varphi} Spec_m(B)$; this map sends a maximal ideal $\mathfrak{m}\subset A$ to its full preimage $\varphi^{*-1}(\mathfrak{m}) \subset B$.*

Proof. A pull back homomorphism of a regular morphism $Spec_m(A) \xrightarrow{\varphi} Spec_m(B)$ takes $\varphi^*(\mathfrak{m}_{\varphi(p)}) = \mathfrak{m}_p$ for any $p\in Spec_m(A)$, because of $f(\varphi(p)) = 0 \iff \varphi^* f(p) = 0$. So, if φ exists, then it has to send $\mathfrak{m} \mapsto \varphi^{*-1}(\mathfrak{m})$. On the other hand, this rule really presents a regular morphism. Indeed, $\varphi^*(1) = 1$ implies that $\varphi^{*-1}(\mathfrak{m}) \subset B$ is a proper maximal ideal, because of $B/\varphi^{*-1}(\mathfrak{m}) = \mathrm{im}\,(\varphi^*)/(\mathfrak{m}\cap \mathrm{im}\,(\varphi^*)) \simeq k$. So, φ is well defined map of sets. Its pull back homomorphism sends a function $b\in B$, which takes a maximal ideal $\mathfrak{q}\in B$ to $b\,(\mathrm{mod}\,\mathfrak{q}) \in k$, to the function $\mathfrak{m} \longmapsto b\,(\mathrm{mod}\,\varphi^{*-1}(\mathfrak{m})) = \varphi^*(b)\,(\mathrm{mod}\,\mathfrak{m}) \in k$, which coincides with the function $\varphi^*(b)$ on $Spec_m(A)$. □

<u>Geometric schemes.</u> Let A be an arbitrary algebra over a field k. All its nilpotents form an ideal $\mathfrak{n}(A) \subset A$ contained in any proper maximal ideal $\mathfrak{m} \subset A$, because $A/\mathfrak{m} = k$ doesn't have nilpotents. So, a factor algebra $A_{\mathrm{red}} \stackrel{\mathrm{def}}{=} A/\mathfrak{n}$ is reduced and has the same maximal spectrum $X = Spec_m A = Spec_m A_{\mathrm{red}}$. If k is algebraically closed, then the intersection of all maximal ideals in A_{red} is zero, because it consists of all functions vanishing everywhere on the affine algebraic set X. Hence, the intersection of all maximal ideals in A coincides with $\mathfrak{n}(A)$. A maximal spectrum $Spec_m A$ of an arbitrary finitely generated algebra A over an algebraically closed field is called *an affine geometrical scheme*. Intuitively, this is an affine algebraic set $X = Spec_m A_{\mathrm{red}}$ equipped with some extra "nilpotent functions" vanishing everywhere on X. Usually, these nilpotents encode some infinitesimal data on X.

<u>Example 22.8: an intersection of affine algebraic sets $X, Y \subset \mathbb{A}^n$ is defined as $V(I(X)+J(X))$, that is, by an union of all equations</u> for X and Y. If the intersection is nontransversal, a factor algebra $A = k[x_1,\dots,x_n]/(I(X)+J(X))$ is not reduced. Of course, the intersection points are exhausted by an algebraic set $Spec_m A_{\mathrm{red}}$, but a significant information will be lost under the replacement of A by A_{red}. If we want, say, to develop an intersection multiplicities technique, then we have to treat intersections as *schemes* rather than as algebraic sets and investigate the difference between A and A_{red}.

<u>A direct product of affine algebraic sets.</u> Let A, B be finitely generated k-algebras. We write $A\otimes B$ for its tensor product as vector spaces[1] over k. In fact, $A\otimes B$ is an finitely generated k-algebra with the multiplication map $(A\otimes B)\times(A\otimes B) \xrightarrow{(a\otimes b,\alpha\otimes\beta)\longmapsto a\alpha\otimes b\beta} A\otimes B$.

Exercise 22.9. Using the universality of the tensor product, verify the correctness of an algebra structure definition and prove that k-algebra homomorphisms $A \xrightarrow{\alpha:a\mapsto a\otimes 1} A\otimes B \xleftarrow{\beta:b\mapsto 1\otimes b} B$ have the following universal property. For any two homomorphisms $A \xleftarrow{\varphi} C \xrightarrow{\psi} B$ there is a (unique) homomorphism $C \xrightarrow{\varphi\otimes\psi} A\otimes B$ such that $\varphi\otimes\psi = \alpha\circ\varphi = \beta\circ\psi$.

<u>Claim 22.10</u> *A product of affine algebraic sets $X = Spec_m A$, $Y = Spec_m B$ has a natural structure of an affine algebraic set given as $X\times Y = Spec_m(A\otimes B)$.*

Proof. On the level of sets, a bijection $Spec_m(A)\times Spec_m(B) \simeq Spec_m(A\otimes B)$ takes any point $(p,q)\in X\times Y$, that is, a pair of epimorphisms $A \xrightarrow{\mathrm{ev}_p} k$, $B \xrightarrow{\mathrm{ev}_q} k$, that is, a bilinear map $A\times B \longrightarrow k$, to an epimorphism $A\otimes B \longrightarrow k$. Its inverse sends an epimorphism $A\otimes B \xrightarrow{\psi} k$ to a pair of epimorphisms $A \xrightarrow{\psi_1} k$, $B \xrightarrow{\psi_2} k$ given by $\psi_1(f) = \psi(f\otimes 1)$, $\psi_2(g) = \psi(1\otimes g)$. To show that $A\otimes B$ is a reduced, we write $\varphi\in A\otimes B$ as $\sum f_\nu\otimes g_\nu$ with linearly independent $g_\nu\in B$; if it produces a zero function on $X\times Y$, then $\varphi(p,q) = 0\ \forall (p,q)\in X\times Y \Rightarrow \sum f_\nu(p)\,g_\nu = 0$ in $k[Y] \Rightarrow$ all $f_\nu = 0$ in $A \Rightarrow \varphi = 0$ in $A\otimes B$. □

Exercise 22.11. Check that the pull back homomorphisms for the projections $X \xleftarrow{\pi_1} X\times Y \xrightarrow{\pi_2} Y$ coincide with the maps α, β from Exercise 22.9 and prove that for any regular morphisms $X \xrightarrow{\alpha} Z$, $Y \xrightarrow{\beta} Z$ there exists an (unique) regular morphism $X\times Y \xrightarrow{\alpha\times\beta} Z$ such that $\alpha\times\beta = \alpha\circ\pi_1 = \beta\circ\pi_2$.

[1] In general, a finitely generated k-algebra is an infinite dimensional vector space with countable base, that is, consists of all *finite* formal linear combinations $\sum\lambda_\nu e_\nu$ with e_ν belonging to some countable family of linearly independent vectors and $\lambda_\nu\in k$. It is easy to see that the finite dimensional constructions from Lect. 8 do work for such spaces as well. In particular, $A\otimes B$ consists of all *finite* sums $\sum a_\nu\otimes b_\nu$ with $a_\nu\in A$, $b_\nu\in B$ and is uniquely defined by the universal property discussed in Lect. 8; for example, $k[x]\otimes k[y]$ is naturally isomorphic to $k[x,y]$.