Algebraic manifolds.

ZARISKI TOPOLOGY. Any affine algebraic set $X = Spec_m A$ admits a canonical topology whose closed sets are zeros of ideals $I \subset A$, that is, the sets $V(I) = \{x \in X \mid f(x) = 0 \ \ \forall f \in I\}$.

Exercise 23.1. Check that $V(I)$ satisfy the closed set properties, namely, $\varnothing = V(1)$; $X = V(0)$; $\bigcap V(I_\nu) = V(\sum I_\nu)$, where $\sum I_\nu$ consists of all finite sums $\sum f_\nu$ with $f_\nu \in I_\nu$; $V(I) \cup V(J) = V(IJ)$, where IJ is an ideal spaned by all products ab with $a \in I$, $b \in J$.

Since any ideal $I \subset k[X]$ is finitely generated, each closed set may be presented as a finite intersection $V(I) = V(f_1, f_2, \dots, f_m) = \bigcap (f_\nu)_0$ for some $f_1, f_2, \dots, f_m \in k[X]$, where $(f)_0 \stackrel{\text{def}}{=} \{x \in X \mid f(x) = 0\}$. So, any Zariski open set is a finite union of *principal* open sets $\mathcal{D}(f) \stackrel{\text{def}}{=} X \setminus (f)_0 = \{x \in X \mid f(x) \neq 0\}$. Zariski topology has a pure algebraic nature: its properties are far enough from the metric topology standards, because Zariski neighborhoods express rather some divisibility conditions than any distance relations.

Example 23.2: Irreducible sets. A topological space X is called *reducible* if $X = X_1 \cup X_2$ for some proper closed subsets $X_1, X_2 \subset X$. It's a vapid notion for the usual metric topology where everything is reducible. In Zariski topology, the reducibility means an existence of non zero $f_1, f_2 \in k[X]$ such that f_1 vanishes along X_1 and f_2 vanishes along X_2. Since $f_1 f_2$ vanishes everywhere, $f_1 f_2 = 0$ in $k[X]$. So, X is reducible iff $k[X]$ has zero divisors. For example, a hypersurface $\{g(x) = 0\} \subset \mathbb{A}^n$ is irreducible iff g is a power of an irreducible polynomial. Irreducible algebraic sets generalize the prime numbers in arithmetics.

Claim 23.3 *Any affine algebraic set admits a unique finite decomposition $X = \bigcup X_i$, where $X_i \subset X$ are irreducible proper closed subsets such that $X_i \not\subset X_j$ $\forall\, i \neq j$ (they are called irreducible components of X).*

Proof. A decomposition is constructed step by step. If X is reducible, the first step takes $X = Z_1 \cup Z_2$, where $Z_{1,2}$ are proper closed subsets. Let $X = \bigcup Z_\nu$ after n steps. If each Z_ν is irreducible, then any irreducible closed subset $Y \subset X$ is decomposed as $Y = \bigcup (Z_\nu \cap Y)$, that is, for each ν either $Y \cap Z_\nu = \varnothing$ or $Y \subset Z_\nu$. So, if we take away all Z_ν contained in some other Z_μ, then we get the required decomposition and it will be unique. If there are some reducible Z_ν after n steps, $(n+1)$-th step replaces each of them by an union of two proper closed subsets. If this procedure would never stop, then it would produce an infinite chain of strictly embedded subsets $X \supset Y_1 \supset Y_2 \supset \dots$, that is, an infinite chain of strictly increasing ideals $(0) \subset I_1 \subset I_2 \subset \dots$, which is impossible in the Noetherian algebra $k[X]$. □

Example 23.4: "big" open sets. Zariski topology is week and non Hausdorf. For example, Zariski topology on $\mathbb{A}^1$ is discrete: $Z \subset \mathbb{A}^1$ is closed iff it is finite. If X is irreducible, then any open $U_1, U_2 \subset X$ have nonempty intersection, because of $X = (X \setminus U_1) \cup (X \setminus U_2)$ in the contrary case.

Exercise 23.5. Prove that $f = g$ in $k[X]$ as soon as X is irreducible and $f|_U = g|_U$ on a nonempty open $U \subset X$.
HINT. If $U = \mathcal{D}(h)$, then $X = V(h) \cup V(f - g)$.

Example 23.6: Zariski topology on $X \times Y$ is finer than the product of Zariski topologies on X and Y, because the closed $Z \subset X \times Y$ are not exhausted by the products of closed subsets on X, Y. For example, if $X = Y = \mathbb{A}^1$, then any curve, say a hyperbola $V(xy - 1)$, is closed in Zariski topology on $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, whereas the products of closed sets on $\mathbb{A}^1$ are exhausted by finite unions of isolated points and coordinate lines.

Claim 23.7 *A regular morphism $X \xrightarrow{\varphi} Y$ of algebraic sets is continuous in Zariski topology.*

Proof. A preimage $\varphi^{-1}(Z)$ of a closed $Z = V(I) \subset Y$ consists of all $x \in X$ such that $0 = f(\varphi(x)) = \varphi^{*-1} f(x)$ for all $f \in I$. So, it coincides with the zero set of an ideal $\varphi^{*-1}(I) \subset k[X]$. □

LOCALIZATION. Let $U \subset X$ be open and $u \in U$. A function $U \xrightarrow{f} k$ is called *regular* at u if there are $p, q \in k[X]$ such that $q(u) \neq 0$ and $f(x) = p(x)/q(x)$ $\forall\, x \in \mathcal{D}(f) \cap U$. The functions $U \xrightarrow{f} k$ which are regular at any $u \in U$ form a commutative ring denoted by $\mathcal{O}_X(U)$ or by $\Gamma(U, \mathcal{O}_X)$. It is called *a ring of local regular functions* on $U \subset X$.

Claim 23.8 *Let X be irreducible and $h \in k[X]$. Then each $f \in \mathcal{O}_X(\mathcal{D}(h))$ takes $f(x) = r(x)/h^d(x)$ for appropriate $r \in k[X]$, $d \in \mathbb{N}$. In particular, $\mathcal{O}_X(X) = k[X]$ for $h = 1$.*

Proof. Of course, $\varphi(x) = r(x)/h^d(x) \in \mathcal{O}_X(\mathcal{D}(h))$. Conversely, let $f \in \mathcal{O}_X(\mathcal{D}(h))$. Then $\forall\, u \in \mathcal{D}(h)$ there are $p_u, q_u \in k[X]$ such that $q_u(u) \neq 0$ and $f(x) = p_u(x)/q_u(x)$ for all $x \in \mathcal{D}(q_u) \cap \mathcal{D}(h)$. Since all common zeros of q_u belong to $V(h)$, Hilbert's Nullstellensatz implies that there are $g_1, g_2, \dots, g_m \in k[X]$ such that $h^d = \sum q_{u_\nu} g_\nu$ for some $d \in \mathbb{N}$, $u_1, u_2, \dots, u_m \in \mathcal{D}(h)$. On the other hand, $f\, q_{u_\nu}(x) = p_{u_\nu}(x)$ everywhere on $\mathcal{D}(h)$ including $\mathcal{D}(h) \cap V(q_{u_\nu})$. Indeed, let $q_{u_\nu}(x) = 0$ and write $f(x) = p_{u_\nu}/q_{u_\nu} = p_x/q_x$ with $q_x(x) \neq 0$; since this identity holds on the nonempty open set $\mathcal{D}(h\,q_{u_\nu}\,q_x)$, we have $p_{u_\nu}\,q_x = q_{u_\nu}\,p_x$ in $k[X]$ by Exercise 23.5, that is, $p_{u_\nu}(x) = q_{u_\nu}(x)\,p_x(x)/q_x(x) = 0$ as well. So, we can write $f(x)\,h^d(x) = \sum f\,q_{u_\nu}\,g_\nu = \sum p_{u_\nu}\,g_\nu \in k[X]$. □

<u>STRUCTURE SHEAF.</u> Let X be an affine algebraic set. A correspondence $U \longmapsto \mathcal{O}_X(U)$ is called *a structure sheaf* on X and a pair $(X, \mathcal{O}_X)$ is called *an affine algebraic manifold*. If $U = \bigcup W_i$ is an union of open sets, then $U \xrightarrow{f} k$ is regular iff each its restriction $f|_{W_i}$ is regular on W_i. Conversely, a collection of functions $W_i \xrightarrow{f_i} k$ such that $f_i \equiv f_j$ on $W_i \cap W_j$ gives a regular function on $U = \bigcup W_i$. Note that U *is not* an affine algebraic set, because, in general, $\mathcal{O}_X(U)$ is not fininitely generated and there are no bijection between $Spec_m\,\mathcal{O}_X(U)$ and U.

Exercise 23.9. Show that $\mathcal{O}_{\mathbb{A}^n}(\mathcal{D}(f))$ is not finitely generated for any non zero $f \in k[\mathbb{A}^n]$.

Exercise 23.10. Let $U = \mathbb{A}^n \setminus O$ be the complement to the origin. Show that $\mathcal{O}_{\mathbb{A}^n}(U) = k[\mathbb{A}^n]$ for $n \geqslant 2$.
HINT. Use the covering $U = \bigcup \mathcal{D}(x_i)$ and Claim 23.8.

<u>ALGEBRAIC MANIFOLDS.</u> Let X be a topological space. An open subset $U \subset X$ is called *an algebraic affine chart* on X if there exists an affine algebraic set X_U and *a homeomorphism* $X_U \xrightarrow{\varphi_U} U$. Two algebraic charts $X_U \xrightarrow{\varphi_U} U$ and $X_W \xrightarrow{\varphi_W} W$ on X are called *compatible* if their *transition map* $\varphi_{WU} = \varphi_W \circ \varphi_U^{-1}$, which identifies $\varphi_U^{-1}(U \cap W) \subset X_U$ with $\varphi_W^{-1}(U \cap W) \subset X_W$, is a regular isomorphism of algebraic open sets, that is, its pull back $\Gamma\left(\varphi_W^{-1}(U\cap W)\,,\,\mathcal{O}_{X_W}\right) \xrightarrow{\varphi_{WU}^*} \Gamma\left(\varphi_U^{-1}(U\cap W)\,,\,\mathcal{O}_{X_U}\right)$ is a well defined isomorphism of k-algebras. A (finite) open covering $X = \bigcup U_\nu$ by mutually compatible algebraic charts is called a (finite) *algebraic atlas* on X. Two algebraic atlases are called *equivalent* if their union is an algebraic atlas too. A topological space X equipped with a class of equivalent (finite) algebraic atlases is called *an algebraic manifold* (of finite type).

Exercise 23.11. Check that projective spaces and Grassmannians are algebraic manifolds of finite type as well as any zero set of a collection of multihomogeneous polynomials on $\mathbb{P}_{n_1} \times \mathbb{P}_{n_2} \times \cdots \times \mathbb{P}_{n_m}$.

<u>Example 23.12:</u> a direct product $X \times Y$ of algebraic manifolds X, Y is an algebraic manifold too. Its atlas consists of all pairwise products $U \times W$, where $U \subset X$, $W \subset X$ are affine algebraic charts on X, Y.

<u>Example 23.13:</u> separability. The standard atlas on $\mathbb{P}_1$ consists of two charts $\varphi_i : \mathbb{A}_1 \xrightarrow{\sim} U_i \subset \mathbb{P}_1$, $i = 0, 1$, and $\varphi_0^{-1}(U_0 \cap U_1) = \varphi_1^{-1}(U_0 \cap U_1) = \{t \in \mathbb{A}^1 \mid t \neq 0\}$ is the complement to the origin. The charts $U_{0,1} = \mathbb{A}^1$ are glued together along $\mathbb{A}^1 \setminus \{O\}$ via transition map $\varphi_{01} : t \mapsto 1/t$. If we replace it by the identity map $\widetilde{\varphi}_{01} : t \mapsto t$, then we get an other manifold called "$\mathbb{A}^1$ with doubled origin", which looks like ——:——. Such the pathology is known as a *nonseparability*. It has appeared, because the last gluing rule is not "complete" and could be extended from $\mathbb{A}^1 \setminus \{O\}$ to a larger set. This can be formalized as follows. Two inclusions $U_0 \hookleftarrow U_0 \cap U_1 \hookrightarrow U_1$ give an embedding $U_0 \cap U_1 \hookrightarrow U_0 \times U_1$. In the case of $\mathbb{P}_1$, this is an inclusion $(\mathbb{A}^1 \setminus O) \hookrightarrow \mathbb{A}^2$ given as $x = t$; $y = 1/t$; it identifies $U_0 \cap U_1$ with *a closed* subset $V(xy - 1) \subset \mathbb{A}^2 = U_0 \times U_1$. In the second case, the embedding $U_0 \cap U_1 \hookrightarrow U_0 \times U_1 = \mathbb{A}^2$ is given as $x = t$; $y = t$ and has a *nonclosed* image $\Delta \setminus \{(0,0)\}$, where $\Delta = V(x - y) \subset \mathbb{A}^2$ is the diagonal. An algebraic manifold X is called *separable* if an image of the canonical embedding $U \cap W \hookrightarrow U \times W$ is closed for any two affine charts $U, W \subset X$. Since this image is nothing more than an intersection of the diagonal $\Delta \subset X \times X$ with the affine chart $U \times W$ on $X \times X$, a manifold X is separable iff the diagonal $\Delta \subset X \times X$ is closed. For example, $\mathbb{A}^n$ and $\mathbb{P}_n$ are separable, because the diagonals in $\mathbb{A}^n \times \mathbb{A}^n$ and in $\mathbb{P}_n \times \mathbb{P}_n$ are given, accordingly, by the equations $x_i = y_i$ and $x_i y_j = x_j y_i$.

<u>REGULAR FUNCTIONS AND MORPHISMS.</u> Let $U \subset X$ be an open set. A function $U \xrightarrow{f} k$ is called *regular* if each point $u \in U$ has an affine neighborhood $X_W \xrightarrow{\varphi_W} W \ni u$ such that $f \circ \varphi \in \mathcal{O}_{X_W}\left(\varphi^{-1}(U \cap W)\right)$. In other words, a local function on X is regular if it induces a local regular functions on each affine chart. Regular functions $U \longrightarrow k$ form a commutative ring $\mathcal{O}_X(U)$; a correspondence $U \longmapsto \mathcal{O}_X(U)$ is called *a structure sheaf* on X. More generally, a map of algebraic manifolds $X \xrightarrow{\varphi} Y$ is called *a regular morphism* if its pull back gives an algebra homomorphism $\mathcal{O}_Y(U) \xrightarrow{\varphi^*} \mathcal{O}_X(\varphi^{-1}U)$ for any open $U \subset Y$. For example, a set of regular morphisms $X \longrightarrow \mathbb{A}^1$ coincides with $\mathcal{O}_X(X)$. Even more generally, a regular[1] morphism $U \xrightarrow{\varphi} Y$, which is defined only on an open dense $U \subset X$, is called *a rational map* from X to Y.

<u>Example 23.14:</u> a projection $\mathbb{A}^{n+1} \xrightarrow{\pi} \mathbb{P}_n$ sending a point $A \in \mathbb{A}^n$ to the line $(OA) \in \mathbb{P}_n$ is a rational surjection defined on $U = \mathbb{A}^n \setminus O$. In terms of the standard affine chart $\mathbb{A}^n \xrightarrow{\varphi_i} U_i = \{(t_0, t_1, \ldots, t_n) \in \mathbb{P}_n \mid t_i = 1\}$, the pull back homomorphism $\mathcal{O}_{\mathbb{P}_n}(U_i) \xrightarrow{\pi^*} \mathcal{O}_{\mathbb{A}^{n+1}}(\pi^{-1}(U_i))$ sends $f(x_1, x_2, \ldots, x_n) \in k[x_1, \ldots, x_n] = \mathcal{O}_{\mathbb{P}_n}(U_i)$ to the rational function $\widetilde{f}(t_0, t_1, \ldots, t_n) = f(t_0/t_i\,,\,\ldots\,,\,t_{i-1}/t_i\,,\,t_{i+1}/t_i\,,\,\ldots\,,\,t_n/t_i) \in \mathcal{O}_{\mathbb{A}^{n+1}}(\mathcal{D}(t_i)) = \mathcal{O}_{\mathbb{A}^{n+1}}(\pi^{-1}(U_i))$.

Exercise 23.15. Prove that the global regular functions on $\mathbb{P}_n$ are exhausted by the constants: $\mathcal{O}_{\mathbb{P}_n}(\mathbb{P}_n) = k$.
HINT. Example 23.14 and Exercise 23.10 give an injection $\mathcal{O}_{\mathbb{P}_n}(\mathbb{P}_n) \xrightarrow{\pi^*} \mathcal{O}_{\mathbb{A}^{n+1}}(U) \cong k[\mathbb{A}^{n+1}]$ whose image consists of all polynomials taking a constant value along each line through the origin; they are constant, certainly.

[1] as above, the regularity means that the pull back $\mathcal{O}_Y(W) \xrightarrow{\varphi^*} \mathcal{O}_X(\varphi^{-1}(W))$ is a well defined k-algebra homomorphism