Some morphisms.

A REGULAR MORPHISM OF AFFINE MANIFOLDS $X \xrightarrow{\varphi} Y$ defines and is uniquely defined by a k-algebra homomorphism $k[Y] \xrightarrow{\varphi^*} k[X]$, which is decomposed as $k[Y] \twoheadrightarrow k[Y]/\ker(\varphi^*) = \mathrm{im}\,(\varphi^*) \hookrightarrow k[X]$. Since $k[X]$ is reduced, $k[Y]/\ker(\varphi^*)$ is too; so, there is an affine manifold $Z = Spec_m\left(k[Y]/\ker(\varphi^*)\right)$ such that $X \xrightarrow{\varphi} Y$ is decomposed as $X \xrightarrow{\varphi_1} Z \xrightarrow{\varphi_2} Y$, where $k[Y] \xrightarrow{\varphi_2^*} k[Z]$ is surjective, and $k[Z] \xrightarrow{\varphi_1^*} k[X]$ is injective. The injectivity of φ_1 means that there are no non zero functions $f \in k[Z]$ vanishing along $\varphi_1(X)$, that is, $\varphi_1(X) \subset Z$ is dense. The surjectivity of φ_2^* means that φ_2 induces an isomorphism between Z and a closed submanifold $V(I) \subset Y$ given by a radical ideal $I = \ker(\varphi_2^*) = \ker(\varphi) \subset k[Y]$. Geometrically speaking, a Zariski closure $Z = \overline{\varphi(X)} \subset Y$ is an affine algebraic submanifold in Y and induced maps $X \longrightarrow \overline{\varphi(X)}$ and $\overline{\varphi(X)} \hookrightarrow Y$ are regular morphisms. A regular morphism of affine manifolds $X \xrightarrow{\varphi} Y$, where X is irreducible, is called *dominant* if $k[Y] \xrightarrow{\varphi^*} k[X]$ is injective, that is, $\overline{\varphi(X)} = Y$. If X is reducible, then φ is called *dominant* if its restriction onto each irreducible component of X is dominant. A regular morphism of affine manifolds $X \xrightarrow{\varphi} Y$ is called *a closed embedding* if $k[Y] \xrightarrow{\varphi^*} k[X]$ is surjective, that is, $X \simeq V(\ker\varphi^*) \subset Y$. For any regular morphism $X \xrightarrow{\varphi} Y$ the coordinate algebra $k[X]$ can be considered as an algebra over $\varphi^*(k[Y]) = k[\overline{\varphi(X)}] \subset k[X]$; φ is called *finite* if $k[X]$ is finitely generated and integer over $\varphi^*(k[Y])$. In other words, φ is finite if $k[X]$ is finitely generated as $\varphi^*(k[Y])$-module, that is, there are some $f_1, f_2, \dots, f_m \in k[X]$ such that any $h \in k[X]$ can be written as $h = \sum \varphi^*(g_i)\, f_i$ for some $g_i \in k[Y]$.

Claim 24.1 *Let $X \xrightarrow{\varphi} Y$ be a finite morphism of affine manifolds. Then $\varphi(Z)$ is closed for any closed $Z \subset X$ and an induced morphism $Z \xrightarrow{\varphi_Z} \varphi(Z)$ is finite; moreover, if $Z \neq X$, then $\varphi(Z) \neq Y$.*

Proof. Let $I = I(Z) \subset k[X]$ be an ideal of $Z \subset X$. A restriction $Z \xrightarrow{\varphi_Z} Y$ corresponds to the composition $\varphi_Z^* : k[Y] \xrightarrow{\varphi^*} k[X] \longrightarrow k[X]/I$. Since $k[X]$ is finitely generated as an $\varphi^*(k[Y])$-module, $k[Z] = k[X]/I$ is finitely generated as a module over $\varphi^*(k[Y])/(I \cap \varphi^*(k[Y])) = \varphi_Z^*(k[Y]) = k[\overline{\varphi(Z)}]$; so, $Z \longrightarrow \overline{\varphi(Z)}$ is a regular finite morphism. To prove that $\varphi(Z) = \overline{\varphi(Z)}$, let $A = k[Z]$, $B = k[\overline{\varphi(Z)}] \subset A$, and $f_1, f_2, \dots, f_m$ generate A as B-module. If $q \in Spec_m(B) \setminus Spec_m(A)$, then its maximal ideal $\mathfrak{m}_q \subset B$ generates a non proper ideal in A, that is, $A = \mathfrak{m}_q \cdot A$. In particular, $f_i = \sum \beta_{i\nu} f_\nu$ for some $\beta_{i\nu} \in \mathfrak{m}_q$ and the zero homomorphism of B-modules $A \longrightarrow 0$ may be presented in terms of the generator system $\{f_\nu\}$ by a matrix $E - (\beta_{\nu i})$. Hence, a multiplication by $\det(E - (\beta_{ij}))$ annihilates A and $\det(E - (\beta_{ij})) = 0$. But this determinant has a form $1 + \beta$ with $\beta \in \mathfrak{m}_q$ and we get a contradiction: $1 \in \mathfrak{m}_q$. To prove that the restriction $Z \xrightarrow{\varphi_Z} Y$ is non dominant for $Z \neq X$, take any non zero function $f \in I \subset k[X]$ and consider its irreducible integrality equation $f^m + \varphi^*(g_1)\, f^{m-1} + \cdots + \varphi^*(g_m) = 0$, where $g_\nu \in k[Y]$. It should have $\varphi^*(g_m) \neq 0$; on the other side, it shows that $\varphi^*(g_m) \in I$. So, $g_m \neq 0$ lies in the kernel of $k[Y] \xrightarrow{\varphi_Z^*} k[Z]$. $\square$

CLOSED SUBMANIFOLDS. A regular morphism of arbitrary algebraic manifolds $X \xrightarrow{\varphi} Y$ is called *a closed embedding* if $W = \varphi^{-1}(U)$ is an affine chart on X for any affine chart $U \subset Y$ and a restriction $W \xrightarrow{\varphi_W} U$ is a finite morphism of affine manifolds. In this case $\varphi(X) \subset Y$ is called *a closed submanifold* given by *an ideal sheaf* $U \longmapsto \ker(\varphi_W^*)$. Usually, we identify X with $\varphi(X)$ refer X as a closed submanifold in Y as well. Typically, a closed submanifold $X \subset Y$ is presented by a system of ideals $I_U \subset k[U]$ corresponding to affine charts $U \subset Y$ in such a way that the restrictions $I_U|_{U\cap W}$ and $I_W|_{U\cap W}$ coincide in $\mathcal{O}_Y(U \cap W)$ for any pair of charts U, W. One can say that an algebraic manifold X is affine iff it admits a closed embedding into an affine space. Similarly, a manifold X is called *projective* if it admits a closed embedding $X \hookrightarrow \mathbb{P}_m$ for some m. A closed submanifold $X \subset Y$ is separable as soon as Y is, because the diagonal in $X \times X$ is a preimage of the diagonal in $Y \times Y$ under an embedding $X \times X \hookrightarrow Y \times Y$. In particular, any affine or projective manifold is separable and has a finite type.

Example 24.2: Morphism graph. Let $X \xrightarrow{\varphi} Y$ be a regular morphism. A preimage of the diagonal $\Delta \subset Y \times Y$ under an induced morphism $X \times Y \xrightarrow{\varphi \times \mathrm{Id}_Y} Y \times Y$ is called *a graph* of φ and is denoted by Γ_φ. Geometrically, $\Gamma_\varphi = \{(x, f(x)\} \subset X \times Y$. It is closed, if Y is separable. For example, a graph of a regular morphism of affine manifolds $Spec_m(A) \xrightarrow{\varphi} Spec_m(B)$ is given by equations $f \otimes 1 = 1 \otimes \varphi^*(f)$ in $A \otimes B$, where f runs through A.

CLOSED MORPHISMS. A regular morphism $X \xrightarrow{\varphi} Y$ is called *closed* if $\varphi(Z) \subset Y$ is closed for any closed $Z \subset X$. Of course, the closed embeddings are closed. Claim 24.1 says that finite morphisms are closed. By Example 22.4 a projection $\mathbb{P}_m \times \mathbb{A}^n \longrightarrow \mathbb{A}^n$ is closed too.

Corollary 24.3 *Let X be a projective manifold and Y be an arbitrary one. Then a projection $X \times Y \longrightarrow Y$ is closed.*

Proof. Taking an affine chart on Y, we can suppose that Y is affine, that is, $X \times Y$ is a closed submanifold in $\mathbb{P}_m \times \mathbb{A}^n$. So, our projection is just a restriction of the projection $\mathbb{P}_m \times \mathbb{A}^n \longrightarrow \mathbb{A}^n$ on a closed subset. □

Corollary 24.4 *If X is projective and Y is separable, then any morphism $X \xrightarrow{\varphi} Y$ is closed.*

Proof. Since Y is separable, a graph $\Gamma_\varphi \subset X \times Y$ is closed. If $Z \subset X$ is closed, then $Z \times Y$ and $\Gamma_\varphi \cap (Z \times Y)$ are closed in $X \times Y$. But $\varphi(Z)$ is just the projection of $\Gamma_\varphi \cap (Z \times Y) \subset X \times Y$ onto Y. □

Corollary 24.5 *If X is a connected projective manifold, then $\mathcal{O}_X(X) = k$ and any regular morphism to any affine manifold contracts X to a point.*

Proof. Let us identify $k = \mathbb{A}^1$ with an affine chart on $\mathbb{P}_1$ and consider a global regular function $X \xrightarrow{f} k$ as a regular morphism $X \xrightarrow{f} \mathbb{P}_1$. Since $f(X) \subsetneq \mathbb{P}_1$ is closed and connected, it is one point. In particular, if $X \xrightarrow{\varphi} \mathbb{A}^n$ is regular, then each coordinate form $x_i \in k[\mathbb{A}^n]$ takes a constant value along $\varphi(X)$. □

Example 24.6: Blow up of a point $p \in \mathbb{A}^n$. Lines passing through a given point $p \in \mathbb{A}^n$ form a projective space $E \simeq \mathbb{P}_{n-1}$. An incidence graph $\mathcal{B}_p = \{(\ell, q) \in E \times \mathbb{A}^n \mid q \in \ell\}$ is called *a blow up* of $p \in \mathbb{A}^n$. Clearly, $\mathcal{B}_p = \mathbb{A}^n$ for $n = 1$. If $n \geqslant 2$, then a projection $\sigma_p : \mathcal{B}_p \twoheadrightarrow \mathbb{A}^n$ is bijective outside $q = p$, but a preimage $\sigma_p^{-1}(p) \subset \mathcal{B}_p$ coincides with E; it is called *an exceptional divisor*. An other projection $\varrho_E : \mathcal{B}_p \twoheadrightarrow E$ fibers $\mathcal{B}_p$ as *a line bundle* over E whose fiber at $\ell \in E$ coincides with $\ell \subset \mathbb{A}^n$; this line bundle is called *a tautological line bundle* over $\mathbb{P}_{n-1}$. Let us identify $\mathbb{A}^n$ with a standard affine chart $U_0 \subset \mathbb{P}_n$ and take homogeneous coordinates $t = (t_0 : t_1 : \cdots : t_n)$ on $\mathbb{P}_n$ and affine coordinates $x = (x_1, x_2, \dots, x_n)$ on $\mathbb{A}^n$ in such a way that $p = (1 : 0 : \cdots : 0)$, $U_0 = \{\, t = (1 : x_1 : \cdots : x_n)\,\}$, and $L \overset{\text{def}}{=} \mathbb{P}_n \setminus U_0$ is given by $t_0 = 0$. Then we can identify E with L by sending $\ell \longmapsto \overline{\ell} \cap L$, where $\overline{\ell} \subset \mathbb{P}_n$ is the projective closure of $\ell \subset \mathbb{A}_n$. Taking $(t_1 : \cdots : t_n)$ as homogeneous coordinates on $L = E$, we can present $\mathcal{B}_p \subset E \times \mathbb{A}_n$ by the equations $t_i x_j = t_j x_i$ in $(t, x) \in E \times \mathbb{A}_n$.

<u>Finite morphisms.</u> A regular morphism of arbitrary algebraic manifolds $X \xrightarrow{\varphi} Y$ is called *finite* if $W = \varphi^{-1}(U)$ is an affine chart on X for any affine chart $U \subset Y$ and the restriction $W \xrightarrow{\varphi_W} U$ is a finite morphism of affine manifolds. Claim 24.1 shows that any finite morphism is closed and a restriction of a finite morphism onto any closed submanifold $Z \subset X$ is a finite morphism as well.

Example 24.7: a projection of a projective manifold $X \subsetneq \mathbb{P}_n$ from a point $p \notin X$ to a hyperplane $L \not\ni p$ is a finite morphism. To check this, we use the notations from Example 24.6 and take $p = (1 : 0 : \cdots : 0)$, $L = \{t \mid t_0 = 0\}$. Consider an affine chart $\mathbb{A}^{n-1} \subset L$, say sending $u = (u_1, u_2, \dots, u_{n-1}) \in \mathbb{A}^{n-1}$ to $q(u) = (0 : u_1 : \cdots : u_{n-1} : 1) \in L \subset \mathbb{P}_n$. Its preimage under the projection $\pi_L^p : X \longrightarrow L$ is isomorphic to a closed submanifold $\widetilde{X} \subset \mathbb{A}^{n-1} \times \mathbb{P}_1$ whose equations are obtained from the initial homogeneous equations in $t = (t_0 : t_1 : \cdots : t_n)$ for $X \subset \mathbb{P}_n$ by a substitution $t = \vartheta_0\, p + \vartheta_1\, q(u)$, where $(\vartheta_0 : \vartheta_1)$ are homogeneous coordinates along the projective lines $(p, q(u)) \subset \mathbb{P}_n$. More precisely, we put[1] $t_0 = \vartheta_0$, $t_1 = \vartheta_1 u_1$, …, $t_{n-1} = \vartheta_1 u_{n-1}$, $t_n = \vartheta_1$ and write the equations for $\widetilde{X}$ as

$$\alpha_0(u)\, \vartheta_0^m + \alpha_1(u)\, \vartheta_0^{m-1} \vartheta_1 + \cdots + \alpha_m(u) \vartheta_1^m = 0\,. \tag{33}$$

Since $p \notin X$, $\widetilde{X}$ does not intersect a hyperplane $E \subset \mathbb{A}^{n-1} \times \mathbb{P}_1$ given by $\vartheta_1 = 0$. In other words, $\widetilde{X} \subset \mathbb{A}^{n-1} \times \mathbb{A}^1 = \mathbb{A}^n$ is an affine manifold given by the equations $\alpha_0(u)\, \vartheta^m + \alpha_1(u)\, \vartheta^{m-1} + \cdots + \alpha_m(u) = 0$ obtained from (33) by taking $\vartheta = \vartheta_0 / \vartheta_1$. Moreover, the leading coefficients $a_0(u)$ of these equations have no common zeros, because any such a zero $z \in \mathbb{A}^{n-1}$ would produce a point $(z, (1:0)) \in \mathbb{A}^{n-1} \times \mathbb{P}_1$ laying on $\widetilde{X} \cap E$. So, $\alpha_0(u)$ generate a non proper ideal in $k[u_1, \dots, u_{n-1}]$, that is, $I(\widetilde{X}) \subset k[u_1, \dots, u_{n-1}][\vartheta]$ contains an integer equation, say $f(\vartheta) = 0$, with $\alpha_0(u) \equiv 1$. Hence, $k[\widetilde{X}] = k[u_1, \dots, u_{n-1}][\vartheta]/I(\widetilde{X})$ is a finitely generated $k[u_1, \dots, u_{n-1}]$-module, because it is a factor module of $k[u_1, \dots, u_{n-1}][\vartheta]/(f(\vartheta))$ spanned over $k[u_1, \dots, u_{n-1}]$ by a finite collection of initial powers t^ν with $0 \leqslant \nu \leqslant \deg f$.

Exercise 24.8. Check that a composition of finite morphisms is finite and prove that any projective manifold admits a finite surjective morphism onto a projective space.

Corollary 24.9 *Each affine manifold X admits a finite surjective morphism φ onto an affine space.*

Proof. Let $X \subsetneq \mathbb{A}^n$, $\mathbb{A}^n$ be embedded in $\mathbb{P}_n$ as a standard chart U_0, $H_\infty = \mathbb{P}_n \setminus U_0$, and $\overline{X} \subset \mathbb{P}_n$ be a projective closure of X. A projection of $\overline{X}$ from any point $p \in H_\infty \setminus (\overline{X} \cap H_\infty)$ onto any hyperplane $L \not\ni p$ induces a finite morphism from $X = \overline{X} \setminus (\overline{X} \cap H_\infty)$ to $\mathbb{A}^{n-1} = L \setminus (L \cap H_\infty)$. If it is non surjective, we repeat this procedure. □

<u>Families of closed submanifolds.</u> Any regular morphism $X \xrightarrow{\pi} Y$ may be considered as a family of closed submanifolds $X_y = \pi^{-1}(y) \subset X$ parametrized by the points $y \in Y$. If $X \xrightarrow{\pi} Y$, $X' \xrightarrow{\pi'} Y$ are two families with the same base, then a regular morphism $X \xrightarrow{\varphi} X'$ is called *a morphism of families* or *a morphism over* Y if it sends X_y to X'_y for any $y \in Y$, that is, if $\pi = \pi' \circ \varphi$. A family $X \xrightarrow{\pi} Y$ is called *constant* or *trivial* if it is isomorphic over Y to a direct product $X_0 \times Y \xrightarrow{\pi_Y} Y$ for some manifold X_0.

[1] geometrically, we consider the blow up of $p \in \mathbb{P}_n$, that is, $\overline{\mathcal{B}}_p = \{(q, t) \in L \times \mathbb{P}_n \mid t \in (pq)\}$, which coincides with the projective closure of $\mathcal{B}_p$ from Example 24.6. Then we lift X along the projection $\overline{\mathcal{B}}_p \xrightarrow{\sigma_p} \mathbb{P}_n$, isomorphic outside p, and intersect it with a preimage of the affine chart $\mathbb{A}^{n-1} \subset L$ along the projection $\overline{\mathcal{B}}_p \xrightarrow{\varrho_L} L$.