Projective spaces: projections and linear isomorphisms.

PROJECTIONS. For any two complementary projective subspaces $L_1, L_2 \subset \mathbb{P}_n$ *a projection*

$$\pi_{L_2}^{L_1} : (\mathbb{P}_n \setminus L_1) \longrightarrow L_2$$

is the map that sends any point $q \in L_2$ to itself and any point $p \in \mathbb{P}_n \setminus (L_1 \sqcup L_2)$ to $\ell \cap L_2$, where ℓ is the unique line passing through p and crossing both L_1 and L_2 (like in Lemma 2.5).

In the homogeneous coordinates $(x_0 : x_1 : \cdots : x_n)$ such that $(x_0 : x_1 : \cdots : x_m)$ are the coordinates in L_1 and $(x_{m+1} : x_{m+2} : \ldots : x_n)$ are the coordinates in L_2, the projection $\pi_{L_2}^{L_1}$ is nothing more then putting $x_\nu = 0$ for $0 \leqslant \nu \leqslant (m+1)$.

Example 3.1: projection of a conic onto a line. The projection

$$\pi_\ell^p : Q \longrightarrow \ell$$

of the conic $Q = \{x_0^2 + x_1^2 = x_2^2\} \subset \mathbb{P}_2$ from the point $p = (1 : 0 : 1) \in Q$ onto the line $\ell = \{x_2 = 0\}$ is bijective. Indeed, if we are siting in the standard affine chart U_2, where $x_2 = 1$, it looks like in fig. 3.1. We see that the pencil of all lines passing through p is 1-1 parametrized by the points $t \in \ell$, because any such line (pt) intersects Q exactly in one more point $q = q(t)$ in addition to p (the tangent line at p, which is given by $x_0 = x_2$, crosses ℓ at the point $t = (0 : 1 : 0)$ that corresponds to $q(t) = p$ itself and lays at the infinity on fig. 3.1). Moreover, this bijection is *birational*, i. e. the corresponding $(q_0 : q_1 : q_2) \in Q$ and $(0 : t_1 : t_2) \in L$ are *rational* algebraic functions of each other. Namely,

$$(t_1 : t_2) = (\, q_1 : (q_2 - q_0)\,)$$
$$(q_0 : q_1 : q_2) = (\, (t_1^2 - t_2^2) : 2\,t_1 t_2 : (t_1^2 + t_2^2)\,)$$

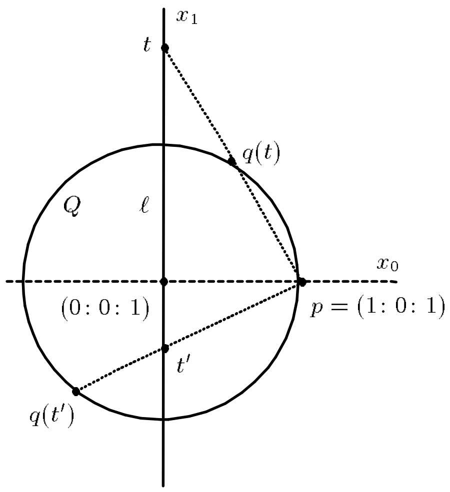

fig. 3.1. Projecting a conic.

Exercise 3.2. Check the above formulas and note that the second of them gives the full list of the Pythagorean triples[1] $(q_0 : q_1 : q_2)$ when (t_1, t_2) runs through $\mathbb{N} \times \mathbb{N}$.

MATRIX NOTATIONS FOR LINEAR MAPS. Let $Hom(U, W)$ be the space of all linear maps from n-dimensional vector space U to m-dimensional vector space W. Denote by $Mat_{m\times n}(k)$ the space of matrices with m columns, n rows, and entries in the field k. Any pair of basises

$$\{u_1, u_2, \ldots, u_n\} \subset U \qquad \text{and} \qquad \{w_1, w_2, \ldots, w_m\} \subset W$$

presents an isomorphism $Hom(U, W) \xrightarrow{\sim} Mat_{m\times n}(k)$ that sends an operator $U \xrightarrow{\alpha} W$ to the matrix $A = (a_{ij})$ such that its j-th column consists of m coordinates of the vector $\alpha(u_j) \in W$ with respect to the basis $\{w_1, w_2, \ldots, w_m\}$,i. e.

$$\alpha(u_j) = \sum_{\nu=1}^{m} a_{\nu j}\, w_\nu$$

or shortly $(\alpha(u_1), \alpha(u_2), \ldots, \alpha(u_n)) = (w_1, w_2, \ldots, w_m) \cdot A$. Let ${}^t z$ and ${}^t y$ be the transposed[2] coordinate rows for $u = (z_1, z_2, \ldots, z_m) \in U$ and $w = \alpha(u) = (y_1, y_2, \ldots, y_n) \in W$. Then ${}^t z = A\,{}^t y$, because of

$$(w_1, w_2, \ldots, w_m) \cdot {}^t z = w = \alpha(u) = \alpha((u_1, u_2, \ldots, u_n) \cdot {}^t y) =$$
$$= (\alpha(u_1), \alpha(u_2), \ldots, \alpha(u_n)) \cdot {}^t y = (w_1, w_2, \ldots, w_m) \cdot (\,A\,{}^t y\,)$$

LINEAR PROJECTIVE TRANSFORMATIONS. If $\dim U = \dim W = (n+1)$, then any linear isomorphism $U \xrightarrow[\sim]{\alpha} W$ induces the bijection $\mathbb{P}(U) \xrightarrow{\sim} \mathbb{P}(W)$, which is called *the projective linear transformation* or *the linear isomorphism.*

[1] i. e. the right triangles with integer side lengths
[2] i. e. arranged in columns

A point set $\{p_1, p_2, \dots, p_m\} \subset \mathbb{P}_n$ is called *linearly general*, if any $(n+1)$ of p_i don't lay together in any hyperplane $\mathbb{P}_{n-1} \subset \mathbb{P}_n$. Equivalently, the points $\{p_i\}$ are linearly general in $\mathbb{P}_n = \mathbb{P}(V)$ iff $(n+1)$ vectors representing any $(n+1)$ of them always form a basis of V.

Lemma 3.3 *For any two linearly general collections of $(n+2)$ points*

$$\{p_0, p_1, \dots, p_{n+1}\} \subset \mathbb{P}_n = \mathbb{P}(U) \qquad \text{and} \qquad \{q_0, q_1, \dots, q_{n+1}\} \subset \mathbb{P}_n = \mathbb{P}(W)$$

there exists a unique up to proportionality linear isomorphism $V \xrightarrow{\alpha} W$ such that $\alpha(p_i) = q_i\ \forall i$. In particular, two matrices give the same projective linear transformation iff they are proportional.

Proof. Fix some vectors u_i and w_i representing the points p_i and q_i. By the linear generality, we can take $\{u_1, u_2, \dots, u_{n+1}\}$ and $\{w_1, w_2, \dots, w_{n+1}\}$ as the basises in U and W and identify a map $\alpha \in Hom(U, W)$ by the square matrix in these basises. Then, $\alpha(p_i) = q_i$ for $1 \leqslant i \leqslant (n+1)$ iff the matrix A of α is diagonal, say with $(d_1, d_2, \dots, d_{n+1})$ on the main diagonal. Now consider the first vectors u_0 and w_0. Again by the linear generality, all the coordinates of $u_0 = (x_1, x_2, \dots, x_{n+1})$ (in the basis $\{u_i\}_{1\leqslant i \leqslant (n+1)}$) and all the coordinates of $w_0 = (y_1, y_2, \dots, y_{n+1})$ (in the basis $\{w_i\}_{1\leqslant i\leqslant (n+1)}$) are no zero. Since the coordinates of $\alpha(u_0)$ are proportional to the ones of w_0, we have $y_i : y_j = (d_i x_i) : (d_j x_j)\ \forall\, i, j$. Hence, all d_i are uniquely recovered from just one of them, say d_1, as $d_i = d_1 \cdot (y_i x_1) : (x_i y_1)$. $\square$

Exercise 3.4. Let ℓ_1 and ℓ_2 be two lines on $\mathbb{P}_2$. Fix any point p outside $\ell_1 \cup \ell_2$ and consider the a $\ell_1 \xrightarrow{\gamma_p} \ell_2$ that sends $t \in \ell_1$ to the crossing point of ℓ_2 and the line (tp). Check that γ_p is the linear isomorphism.

THE LINEAR PROJECTIVE GROUP. If $U = W = V$, then all linear isomorphisms $V \longrightarrow V$ form the group. It is called *a general linear group* and denoted by $GL(V)$. This group acts on $\mathbb{P}(V)$. By Lemma 3.3, the kernel of this action coincides with the subgroup of the scalar homotheties $H \subset GL(V)$. Hence, the group of all projective linear automorphisms of $\mathbb{P}(V)$ equals to the factor group $GL(V)/H$, which is denoted by $PGL(V)$ and called *a projective linear group.*

As soon as some basis $\{e_0, e_1, \dots, e_n\} \subset V$ is fixed, we can identify $GL(V)$ with the group $GL_{n+1}(k) \subset Mat_{n+1}(k)$ of all non degenerated square matrices. Under this identification the homotheties go to the scalar matrices and $PGL(V)$ turns into the group

$$PGL_{n+1}(k) \stackrel{\text{def}}{=} GL_{n+1}(k)/\{\text{scalar diagonal matrices } \lambda\, E\}$$

of all non degenerate square matrices considered up to proportionality.

Example 3.5: linear-fractional group The group $PGL_2(k)$ consists of all square matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which have $ad - bc \neq 0$ and are considered up to proportionality. It acts on $\mathbb{P}_1$ via

$$(x_0 : x_1) \xmapsto{A} (\,(ax_0 + bx_1) : (cx_0 + dx_1)\,)$$

In terms of the standard affine chart $U_0 \simeq \mathbb{A}_1$ with the affine coordinate $t = x_1/x_0$, this action is given by

$$A\, :\, t \longmapsto \frac{ct + d}{at + b}$$

Exercise 3.6. Check that $(AB)(t) = A(B(t))$.

Exercise 3.7. For any 3 different points $p, q, r \in U_0$ write down the unique linear-fractional transformation α such that $\alpha(p) = 0$, $\alpha(q) = 1$, and $\alpha(r) = \infty$.

HINT. It has to be of the form $t \xmapsto{\alpha} \vartheta \dfrac{t-p}{t-r}$, where $\vartheta \in k$.

Proposition 3.8 *Let $\mathbb{P}_1 \xrightarrow{\varphi} \mathbb{P}_1$ be a bijection over an algebraically closed field. If everywhere except for a finite number of points φ is given by the formula $\varphi(x_0 : x_1) = (f_0(x_0, x_1) : f_1(x_0, x_1))$, where f_i are rational algebraic functions, then $\varphi \in PGL_2(k)$.*

Proof. To give a well defined map, the functions f_i have to be homogeneous of the same degree, moreover, after appropriate elimination we can suppose that f_i are *coprime polynomials*. The map $U_0 \longrightarrow U_0$ induced by φ sends $t = x_1/x_0$ to $\lambda = f_1(1,t)/f_0(1,t)$. Since the equation $\lambda\, f_0(1,t) - f_1(1,t) = 0$ has exactly $\deg f_i$ distinct roots t for almost all[3] λ, the bijection is possible only for linear f_i. $\square$

[3] except for a finite number of values where *the discriminant* vanishes