Projective quadrics: the basics.

QUADRATIC AND BILINEAR FORMS. The zero set $Q = (q)_0 \subset \mathbb{P}(V)$ of a non zero quadratic form $q \in S^2V^*$ is called *a projective quadric*. If $2 \neq 0$ in k ([1]), then in homogeneous coordinates any quadratic form can be written as

$$q(x) = \sum_{i,j} a_{ij}\, x_i x_j = x \cdot A \cdot {}^t x\ ,$$

where $x = (x_0, x_1, \ldots, x_n)$ is the coordinate row, ${}^t x$ is its transposed version, and $A = (a_{ij})$ is a *symmetric* matrix over k, called *a Gram matrix* of q. Hence, there exists a unique bilinear form $\widetilde{q}(u, w)$ on $V \times V$ such that $q(x) = \widetilde{q}(x, x)$. It can be calculated by the following pairwise equivalent ways

$$\widetilde{q}(x, y) = \sum_{ij} a_{ij}\, x_i y_j = x \cdot A \cdot {}^t y = \frac{1}{2} \sum_i y_i \frac{\partial q(x)}{\partial x_i} =$$

$$= \frac{1}{2}\Big(q(x + y) - q(x) - q(y)\Big) = \frac{1}{4}\Big(q(x + y) - q(x - y)\Big)$$

The form $\widehat{q}$ is called *a polarization* of q and gives a kind of scalar product on V. The Gram matrix elements are just the scalar products $a_{ij} = \widetilde{q}(e_i, e_j)$ of basic vectors $\{e_i\}$. Under a base change, say by $(e'_0, e'_1, \ldots, e'_n) = (e_0, e_1, \ldots, e_n)\cdot C$, the old Gram matrix A will be changed by the new one $A' = {}^t C\cdot A\cdot C$. So, *a Gram determinant* $\det q \overset{\text{def}}{=} \det A \in k/k^2$, which is considered up to multiplications by the squares from k, doesn't depend on the choice of base.

A quadric is called *non singular* if $\det q \neq 0$. Otherwise, it is called *singular*. Two quadrics are called *isomorphic* or *projectively equivalent* if they are produced from each other by a projective automorphism (i. e. by a base change).

Theorem 4.1([2]) *Any quadratic form on V has a diagonal Gram matrix in some base.*

Proof. If $q \equiv 0$, then its Gram matrix is diagonal. If not, then $q(v) = \widetilde{q}(v, v) \neq 0$ for some $v \in V$. Take it as the first vector of the base being constructed. Note that $V = L \oplus L^\perp$, where L is one dimensional subspace spaned by v and $L^\perp = \{\, w \in V \mid \widetilde{q}(v, w) = 0 \,\}$ is its orthogonal. Indeed, any $u \in V$ is uniquely decomposed as $u = \lambda v + v^\perp$ with $v^\perp \in L^\perp$. Namely, $\lambda = \widetilde{q}(v, u)/\widetilde{q}(v, v)$ and $v^\perp = u - (\widetilde{q}(v, u)/\widetilde{q}(v, v)) \cdot v$. Now, repeat these arguments to $L^\perp$ to find the next base vector, and so on. □

Corollary 4.2 *Over an algebraically closed field any quadric may be written as $\sum x_i^2 = 0$ in some coordinate system; in particular all non singular quadrics are projectively equivalent.*

Proof. A diagonal Gram matrix turns into the identity one after the base change $e_i \longmapsto e_i/\sqrt{q(e_i)}$. □

Example 4.3: A quadric in $\mathbb{P}_1$, by Lagrange theorem, may be written either as $a\, x_0 + b\, x_1 = 0$, or as $a\, x_0^2 = 0$. The second is singular (i. e. $\det q = 0$) and is called *a double point*. It consists of just one point $(0\colon 1)$, which has "a multiplicity 2" in any reasonable sense. The first is non singular and either is empty or consists of two distinct points. If $-\det q \equiv -b/a \pmod{k^2}$ is not a square in k, then $Q = \varnothing$. Note that this is impossible over an algebraically closed field. If $-\det q$ is a square, say δ^2, then $Q = \{\, (-\delta/a : 1)\, ,\, (\delta/a : 1) \,\}$.

QUADRIC AND LINE. By the above example, for any quadric Q and a line ℓ there are exactly 4 possibilities: either $\ell \subset Q$, or $\ell \cap Q$ consists of 2 distinct points, or $\ell \cap Q$ is a double point, or $\ell \cap Q = \varnothing$. Moreover, the last case is impossible if k is algebraically closed. A line, which lays on Q or crosses it via the double point is called *a tangent line*.

CORRELATIONS. Any quadratic form q on V induces the linear map $V \xrightarrow{\widehat{q}} V^*$ that sends a vector $v \in V$ to the linear form

$$\widehat{q}(v)\ :\ w \longmapsto \widetilde{q}(v, w)$$

The map $\widehat{q}$ is called *a correlation* of q. Its matrix written in dual bases $\{e_i\} \subset V$, $\{x_i\} \subset V^*$ coincides with the Gram matrix A. In particular, q is non singular iff $\widehat{q}$ is isomorphic. The space

$$\ker(q) \overset{\text{def}}{=} \ker \widehat{q} = \{\, v \in V \mid \widetilde{q}(v, w) = 0\ \forall\, w \in V\ \}$$

[1] WARNING: in this lecture we always suppose that $\operatorname{char} k \neq 2$

[2] It is known as the *Lagrange theorem*

is called *a kernel* of q. Its projectivization $Sing\,(Q) \stackrel{\text{def}}{=} \mathbb{P}(\ker q) \subset \mathbb{P}(V)$ is called *a vertex space* of $Q = (q)_0$ and $\operatorname{codim}_{\mathbb{P}(V)} Sing\,(Q)$ is called *a corank* of Q.

Theorem 4.4 *The intersection $Q' = L \cap Q$ is non singular for any projective subspace $L \subset \mathbb{P}(V)$ complementary to $Sing(Q)$; moreover, Q is the cone over Q' with the vertex space $Sing(Q)$, i. e. Q is the union of all lines crossing both Q' and $Sing(Q)$.*

Proof. Take any direct decomposition $V = \ker q \oplus U$ and let $L = \mathbb{P}(U)$. If $u \in U$ satisfy $\widetilde{q}(u, u') = 0 \quad \forall\, u' \in U$, then automatically $\widetilde{q}(u, v) = 0 \quad \forall\, v \in V$ and $u = 0$, because of $\ker q \cap U = 0$. Since $Q' = Q \cap L$ is given by the restriction $q|_U$, it is non singular. Further, for any line $\ell = \mathbb{P}(W)$ such that $\dim W \cap \ker Q = 1$ we have $\dim W \cap U = 1$ and $\operatorname{cork} q|_U \geqslant 1$. So, if $\ell \cap Sing(Q) = \{p\}$ is just one point, then $\ell \cap L \neq \varnothing$ and either $\ell \subset Q$ or $\ell \cap Q\{p\}$. That's all we need. $\square$

Corollary 4.5 *A quadric $Q \subset \mathbb{P}_n$ over an algebraically closed field is uniquely up to an isomorphism defined by its corank; it may be equal to* $0, 1, \ldots, n$.

Proof. Corank is the number of diagonal zeros in the diagonal Gram matrix. $\square$

<u>TANGENT SPACE</u> T_pQ to a quadric $Q = (q)_0$ at a point $p \in Q$ is defined as the union of all tangent lines passing through p.

Lemma 4.6 *Let p and p' be distinct points and $p \in Q = (q)_0$. The line $\ell = (p\,p')$ is tangent to Q iff $\widetilde{q}(p, p') = 0$, i. e. iff p and p' are orthogonal with respect to polarization of q.*

Proof. Take some vectors u, u' representing p and p'. Then $\ell = \mathbb{P}(U)$. The restriction $q|_U$ has the Gram matrix $\begin{pmatrix} \widetilde{q}(u,u) & \widetilde{q}(u,u') \\ \widetilde{q}(u',u) & \widetilde{q}(u',u') \end{pmatrix}$. It is singular iff $\widetilde{q}(u, u') = 0$, because of $\widetilde{q}(u, u) = 0$ by the lemma assumption. $\square$

Corollary 4.7 *$p \in Sing(Q)$ iff T_pQ is the whole space, and iff* $\dfrac{\partial q(x)}{\partial x_i}(p) = 0 \ \forall i$. $\square$

Corollary 4.8 *If $p \in (Q \setminus Sing(Q))$, then T_pQ is the codimension 1 hyperplane given by the linear equation $\widetilde{q}(p, x) = 0$ on $x \in \mathbb{P}_n$.* $\square$

Corollary 4.9 *Let $p \notin Q$ and $C \subset \mathbb{P}_n$ be the hyperplane given by the equation $\widetilde{q}(p, x) = 0$ on x. Then $Q \cap L$ consists of all points where Q is crossed by the tangent lines coming from p.* $\square$

<u>POLAR MAPPINGS.</u> The spaces $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ are called *dual* and denoted by $\mathbb{P}_n$ and $\mathbb{P}_n^\times$ when V is not essential. Since any codimension 1 subspace $U \subset V$ is defined by linear form $\xi \in V^*$, which is unique up to proportionality, $\mathbb{P}_n^\times$ is just *the space of hyperplanes* in $\mathbb{P}_n$ and vice versa. If $Q = (q)_0 \subset \mathbb{P}(V)$ is non singular, then the linear isomorphism $\mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V^*)$ induced by the correlation $\widehat{q}$ is called *a polarity* of Q. It sends a point $p \in \mathbb{P}_n$ to the hyperplane $L \subset \mathbb{P}_n$ given by the equation $\widetilde{q}(p, x) = 0$ like in the previous Corollary. L is called *a polar* of p and p is called *a pole* of L with respect to q. So, Q is just the set of all points laying on their own polars. Note that some non singular quadratic forms q can produce *empty* quadrics Q over an non closed field; but their polar mappings will be always visible!

Exercise 4.10. Show that p lays on the polar of q iff q lays on the polar of p (for any pair of distinct points and any polarity).

Exercise 4.11. Consider a circle in the real Euclidean affine plane $\mathbb{R}^2$. How to draw the polar of a point that lays: a) outside b) inside this circle? Describe geometrically the polarity defined by the "imaginary" circle "given" in $\mathbb{R}^2$ by the equation $x^2 + y^2 = -1$.

Proposition 4.12 *Two polaritets coincide iff the corresponding quadratic forms are proportional.*

Proof. This follows from Lemma 3.3 $\square$

Corollary 4.13 *Over an algebraically closed field two quadrics coincide iff their quadratic equations are proportional.*

Proof. Let $Q = Q'$. We can suppose that the quadrics are non singular, because of their equations are not changed under direct summation with the kernel $\ker q = \ker q'$. Non singular case is covered by the above Proposition. $\square$

<u>THE SPACE OF QUADRICS.</u> All the polaritets on $\mathbb{P}_n = \mathbb{P}(V)$ are one-to-one parametrized by the points of the projective space

$$\mathbb{P}_{\frac{n(n+3)}{2}} = \mathbb{P}(S^2V^*)\ ,$$

which will be reffered as *a space of quadrics*.