Quadrics: complex plane conics[1].

WHO ARE THE CONICS. A quadric on the complex projective plane is called *a complex projective conic*. Up to an isomorphism, it coincides either with *a double line* $x_0 = 0$ of corank 2 or with *a reducible conic*[2] $x_0^2 + x_1^2 = 0$ of 1 or with the non singular conic $x_0^2 + x_1^2 + x_2^2 = 0$ of corank 0. The space of all conics in $\mathbb{P} - 2 = \mathbb{P}(V)$ is $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$.

Proposition 5.1 *Any 5 points in $\mathbb{P}_2$ lay on some conic. This conic is non singular, If no 3 of the points are collinear.*

Proof. Since the equation $q(p) = 0$ is linear in q for a given p, the conics passing through $p \in \mathbb{P}_2$ form the hyperplane in $\mathbb{P}_5 = \mathbb{P}(S^2V^*)$, where any 5 hyperplanes have non zero intersection. □

THE STANDARD MODEL FOR A NON SINGULAR CONIC. For a 2-dimensional vector space U *a Veronese map* $U^* \xrightarrow{v} S^2U^*$ sends a linear form ξ to its square ξ^2 and induces the embedding

$$\mathbb{P}(U^*) = \mathbb{P}_1 \overset{v}{\hookrightarrow} \mathbb{P}_2 = \mathbb{P}(S^2U^*)\ .$$

Identify $\mathbb{P}(S^2U^*)$ with the space of quadrics on $\mathbb{P}_1 = \mathbb{P}(U)$. Then the Veronese embeddinq gives a bijection between the points of $\mathbb{P}_1$ and the singular quadrics on $\mathbb{P}_1$, which are just the double points. Hence, its image $Q_V \overset{\text{def}}{=} \operatorname{im}(v) = \{\, q \in S^2U^* \mid \det q = 0 \,\}$ is the quadric; it is called *a Veronese quadric*.

If (x_0, x_1) are the coordinates in U and the base $\{x_0^2,\, 2\,x_0x_1,\, x_1^2\}$ in S^2U^* is taken, then a quadric $q(x) = q_0\, x_0^2 + 2\, q_1\, x_0x_1 + q_2\, x_1^2 \in S^2U^*$ is coordinated by $(q_0 \colon q_1 \colon q_2)$ and a linear form $\xi(x) = t_0x_0 + t_1x_1 \in U^*$ — by $(t_0 \colon t_1)$. Veronese sends $(t_0x_0 + t_1x_1) \longmapsto (t_0x_0 + t_1x_1)^2 = t_0^2\, x_0^2 + 2\, t_0t_1\, x_0x_1 + t_1^2\, x_1^2$. So, Q_V has the equation $\det(q) = q_0q_2 - q_1^2 = 0$ and is parametrized by $(q_0 \colon q_1 \colon q_2) = (t_0^2 : t_0t_1 : t_1^2)$.

Note that any non singular conic $Q \subset \mathbb{P}_2$ can be identified with Q_V by an appropriate base choice.

Proposition 5.2 *Two distinct non singular conics cann't have more then 4 common points.*

Proof. Parametrize the first conic by $x = v(t_0, t_1)$ via Veroneze. If the second conic is given by the quadratic equation $q(x) = 0$, then the t-coordinates of the intersection points are given by the 4-th degree equation $q(v(t)) = 0$. □

Corollary 5.3 *The conic from Proposition 5.1 is unique, if no 3 of points are collinear.* □

DUALITY. For any $0 \leqslant m \leqslant (n-1)$ there is a canonical bijection between the m-dimensional projective subspaces in $\mathbb{P}_n = \mathbb{P}(V)$ and the $(n-1-m)$-dimensional ones in $\mathbb{P}_n^\times \overset{\text{def}}{=} \mathbb{P}(V^*)$. It sends a subspace $L = \mathbb{P}(U)$ to the subspace $L^\times \overset{\text{def}}{=} \mathbb{P}(\operatorname{Ann}(U))$, where $\operatorname{Ann}(U) \overset{\text{def}}{=} \{\, \xi \in V^* \mid \xi(u) = 0 \quad \forall\, u \in U \,\}$ is *an anihilator* of U. Since under the natural identification $V^{**} = V$ we have $\operatorname{Ann}\operatorname{Ann} U = \{\, v \in V \mid \xi(v) = 0 \quad \forall\, \xi \in \operatorname{Ann} U \,\} = U$, then $L^{\times\times} = L$. The correspondence $L \leftrightarrow L^\times$ is called *a projective duality*. It inverts the inclusions[3] and the linear incidence: $L_1, L_2, \ldots, L_m$ are contained in some m-dimensional subspace L iff $L_1^\times, L_2^\times, \ldots, L_m^\times$ contain some m-dimensional subspace $L^\times$ ([4]). The projective duality translates the geometry on $\mathbb{P}_n$ to the one on $\mathbb{P}_n^\times$ and back. For example, in $\mathbb{P}_2$-case we have the following dictionary:

a line $\ell \subset \mathbb{P}_2$	$\longleftrightarrow$	a point $\ell^\times \in \mathbb{P}_2^\times$
the points p of the above line ℓ	$\longleftrightarrow$	the lines $p^\times$ passing through the above point $\ell^\times$
the line passing through two points $p_1, p_2 \in \mathbb{P}_2$	$\longleftrightarrow$	the intersection point for two lines $p_1^\times, p_2^\times \subset \mathbb{P}_2^\times$
the points p of some conic Q	$\longleftrightarrow$	the tangent lines $p^\times$ of some conic $Q^\times$
the tangent lines ℓ of Q	$\longleftrightarrow$	the points $\ell^\times$ of $Q^\times$

Exercise 5.4. Explain the last two items by proving that the tangent spaces of a non singular quadric $Q \subset \mathbb{P}_n$ correspond to the points of some non singular quadric $Q^\times \subset \mathbb{P}_n^\times$. Show also that Q and $Q^\times$ have inverse Gram matrices in dual bases of V and V^*.

HINT. If $\xi = (\xi_0, \xi_1, \ldots, \xi_n) \in V^*$ and $x = (x_0, x_1, \ldots, x_n) \in V$ in dual coordinate systems, then $\xi(x) = \xi \cdot {}^t x$. Let Q be given by $x \cdot A \cdot {}^t x = 0$. Then $T_xQ = \mathbb{P}(\operatorname{Ann}\xi) \iff \xi = x \cdot A \iff x = \xi \cdot A^{-1}$. Hence,

$$x \in Q \iff x \cdot A \cdot {}^t x = 0 \iff (\xi \cdot A^{-1}) \cdot A \cdot {}^t(\xi \cdot A^{-1}) = 0 \iff \xi \cdot A^{-1}\, {}^t\xi = 0 \iff \xi \in Q^\times\ ,$$

where $Q^\times$ is given by the Gram matrix A^{-1}.

[1] in this lecture $k = \mathbb{C}$ or k is "algebraically closed field of zero characteristic" (what is essentially the same)
[2] i. e. a pair of crossing lines
[3] i. e. $L_1 \subset L_2 \iff L_1^\times \supset L_2^\times$
[4] for example, 3 points are collinear iff the dual 3 hyperplanes have common codimension 2 subspace

Corollary 5.5 *Any 5 lines without triple intersections in $\mathbb{P}_2$ are tangent to a unique non singular conic.*

Proof. This assertion is projectively dual to Corollary 5.3 and Proposition 5.1. Namely, let $\ell_i \in \mathbb{P}_2$ be the given lines. There exists a unique conic $Q^\times$ passing through the 5 points $\ell_i^\times \in \mathbb{P}_2^\times$. Then $\ell_i = \ell_i^{\times\times}$ are tangent to the dual conic $Q = Q^{\times\times} \subset \mathbb{P}_2$. $\square$

Examples of projective linear isomorphisms $\mathbb{P}_1 \longrightarrow \mathbb{P}_1$. Consider the projection of a non singular conic to a line ℓ $\pi_\ell^p : Q \longrightarrow \ell$ Q from a point $p \in Q$ and extend it to the bijection $Q \xrightarrow{\sim} \ell$ by sending $p \longmapsto \ell \cap T_pQ$.

Exercise 5.6. Show that this bijection is given by some rational algebraic functions, which express coordinates of the corresponding points through each other.

Hint. To express the coordinates of $t = \pi_\ell^p(x)$ through x, you take special coordinates where ℓ is given by $x_0 = 0$ and $p = (1\colon 0\colon 0)$; then $t = (0\colon x_1 : x_2)$. In order to get the inverse expression, only two skills are quite enough: linear equations solving and finding the second root for a quadratic equation with the first root known. The both procedures have a rational output.

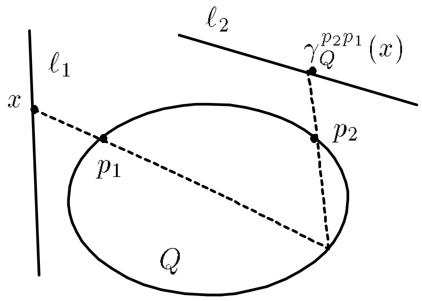

fig. 5.1. Composing projections.

So, if ℓ_1, ℓ_2 are two lines and p_1, p_2 are two distinct points on some non singular conic Q, then the composition $\gamma_Q^{p_2p_1} \stackrel{\text{def}}{=} \pi_{\ell_2}^{p_2} \circ \left(\pi_{\ell_1}^{p_1}\right)^{-1}$ gives the projective linear isomorphism $\ell_1 \xrightarrow{\sim} \ell_2$ (see fig. 5.1).

Lemma 5.7 *Any projective linear isomorphism $\ell_1 \xrightarrow{\gamma} \ell_2$ can be presented*[5] *as $\gamma_Q^{p_2p_1}$ for some Q and $p_1, p_2 \in Q$.*

Proof. Let γ send $a_1, b_1, c_1 \in \ell_1$ to $a_2, b_2, c_2 \in \ell_2$. Take any p_1 and p_2 such that no 3 of 5 points p_1, p_2, $(a_1\, p_1) \cap (a_2\, p_2)$, $(b_1\, p_1) \cap (b_2\, p_2)$, and $(c_1\, p_1) \cap (c_2\, p_2)$ are collinear (see fig. 5.2). Then $\gamma = \gamma_Q^{p_2p_1}$, where Q is the conic drawn through this 5 points, because the both projective linear isomorphisms have the same action on 3 points a_1, b_1, c_1. $\square$

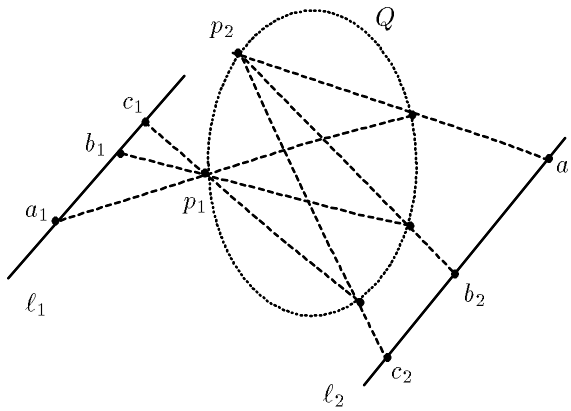

fig. 5.2. Composing projections.

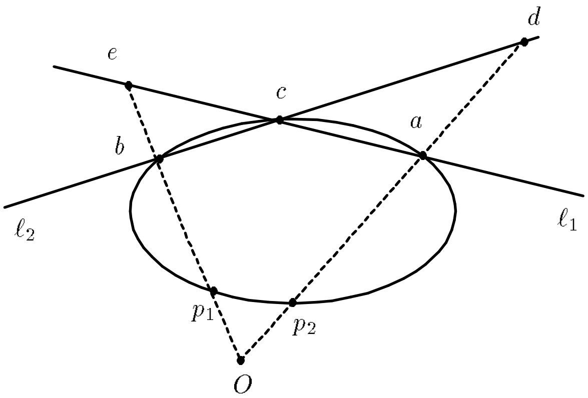

fig. 5.3. Remarkable coincidence.

Remark 5.8 *Let 5 distinct points p_1, p_2, a, b, c lay on a non singular conic Q. Denote the line $(a\, c)$ by ℓ_1, the line $(b\, c)$ by ℓ_2, and the intersection point $(a\, p_2) \cap (b\, p_1)$ by O (see fig. 5.3). Then projective linear isomorphism $\ell_1 \xrightarrow{\gamma_Q^{p_2p_1}} \ell_2$ coincides with the projective linear isomorphism $\ell_1 \xrightarrow{\gamma_O} \ell_2$, which sends $x \in \ell_1$ to $\gamma_O(x) = (x\, O) \cap \ell_2$.*

Proof. The both send $c \longmapsto c$, $a \longmapsto d$, $e \longmapsto b$, where $d = (a\, p_2) \cap \ell_2$ and $e = (b\, p_1) \cap \ell_1$ (see fig. 5.3). $\square$

[5] by many different ways