Quadrics: some plane drawings.

<u>DRAWING A CONIC BY THE RULER.</u> Remark 5.8 is a powerful tool for drawing things by the ruler. For example, it gives an effective method for a dense point set tracing on the conic Q passing through 5 given points $p_1, p_2, \dots, p_5$. Namely, take $\ell_1 = (p_3p_4)$, $\ell_2 = (p_4p_5)$, $O = (p_1p_6) \cap (p_2p_3)$ (see fig. 6.1). Then any line $L \ni O$ gives two intersection points $\ell_1 \cap L$ and $\ell_2 \cap L$. These points are sent to each other by the projective linear isomorphism $\gamma_O = \gamma_Q^{p_2p_1}$. So, if we draw lines through p_1 and $\ell_1 \cap L$, and trough p_2 and $\ell_2 \cap L$, then the intersection point x of these two lines has to lay on Q. On the fig. 6.1 the points x_1, x_2, x_3 are constructed by this way starting from the lines L_1, L_2, L_3 passing through O.

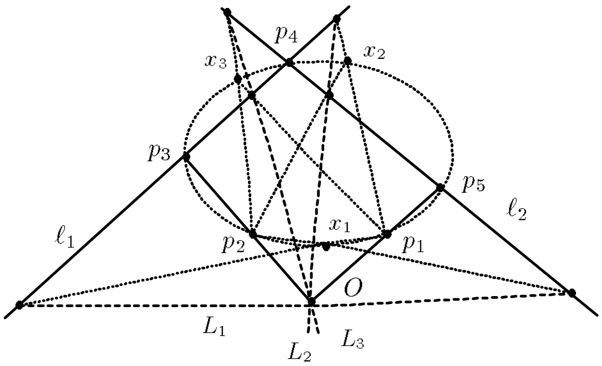

fig. 6.1. Drawing a conic.

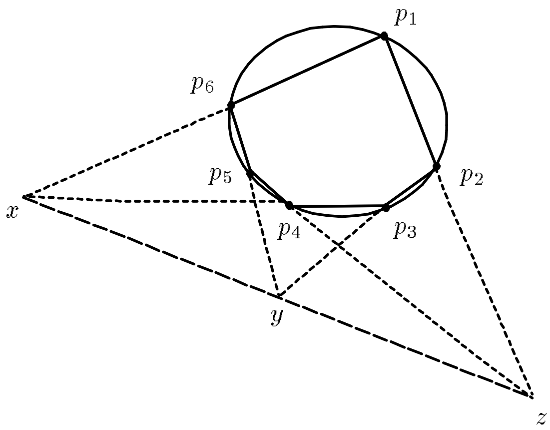

fig. 6.2. An inscribed hexagon.

<u>Corollary 6.1</u>[1] *A hexagon $p_1, p_2, \dots, p_6$ is inscribed into a non singular conic iff the points*[2]*$(p_1p_2) \cap (p_4p_5)$, $(p_2p_3) \cap (p_5p_6)$, $(p_3p_4) \cap (p_6p_1)$ are collinear (see fig. 6.2).*

Proof. Draw the conic Q through 5 of p_i except for p_4 and put $\ell_1 = (p_1p_6)$, $\ell_2 = (p_1p_2)$, $y = (p_5p_6) \cap (p_2p_3)$, $x = (p_3p_4) \cap \ell_1$, $z = (xy) \cap \ell_2$. Then $z \in \ell_2$ is the image of $x \in \ell_1$ under the projective linear isomorphism

$$\gamma_y = \gamma^{p_5p_3} : \ell_1 \xrightarrow{\sim} \ell_2$$

like before. In particular, the intersection point $(p_3p_4) \cap (p_5z)$ lays on Q. Hence, $p_4 \in Q$ iff $p_4 = (p_3p_4) \cap (p_5z)$. □

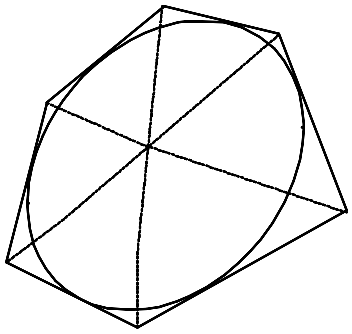

fig. 6.3. A circumscribed hexagon.

<u>Corollary 6.2</u>[3] *A hexagon $p_1, p_2, \dots, p_6$ is circumscribed around a non singular conic iff its main diagonals (p_1p_4), (p_2p_5), (p_3p_6) are intersecting at one point (see fig. 6.3).*

Proof. This is just the projectively dual version of the Pascal theorem. □

<u>LINEAR ISOMORPHISMS OF PENCILS.</u> A family of geometrical figures is referred as *a pencil*, if it is naturally parametrized by the projective line. For example, all lines passing through a given point $p \in \mathbb{P}_2$ form the pencil, because their equations run trough the line $p^\times \in \mathbb{P}_2^\times$ by projective duality. More generally, there are pencils of the hyperplanes $H \subset \mathbb{P}_n$ passing through a given codimension 2 subspace $L \subset \mathbb{P}_n$. They are denoted by $|h - L|$ ("all hyperplanes containing L") or by $L^\times \in \mathbb{P}_n^\times$.

Consider two such pencils, say $L_1^\times$ and $L_2^\times$. Let 3 points $a, b, c \in \mathbb{P}_n \setminus (L_1 \cup L_2)$ be such that 3 hyperplanes from $L_i^\times$ passing through these points are distinct in the both pencils. Then these 3 points define a projective linear isomorphism $L_1^\times \xrightarrow{\gamma_{abc}} L_2^\times$ which sends 3 hyperplanes of the first pencil passing through a, b, c to the corresponding ones from the second pencil.

<u>Example 6.3:</u> projective linear isomorphism between two pencils $p_1^\times$ and $p_2^\times$ on $\mathbb{P}_2$ is given by any 3 points a, b, c such that any 2 of them are not collinear with p_1 or p_2. It sends $(p_1a) \longmapsto (p_2a)$, $(p_1b) \longmapsto (p_2b)$, $(p_1c) \longmapsto (p_2c)$. Let Q be the (unique!) conic passing trough 5 points p_1, p_2, a, b, c. There are two different cases (see fig. 6.4–6.5).

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[2]It is known as *the Pascal theorem*

[2]i. e. 3 intersection points of the line pairs passing through the opposite sides of the hexagon

[3]It is known as *the Pappus theorem*

(A) Elliptic case: Q is non singular, i. e. all 5 points are linearly general. In this case *the incidence graph*[4] of γ_{abc} coincides with Q, because the points of Q give the projective linear isomorphism $p_1^\times \longrightarrow p_2^\times$ that has the same action on a, b, c. Moreover, the above discussions let us draw the line $\gamma_{abc}(\ell_x)$ for a given $(\ell_x \in p_1^\times)$ by the ruler as follows (see fig. 6.4). First mark the point $O = (p_1 b) \cap (p_2 c)$; then find the intersection point $\ell_x \cap (a\,c)$, join it with O by a line, and mark the point where this line crosses $(b\,c)$; then the line $\gamma_{abc}(\ell_x)$ goes through this marked point.

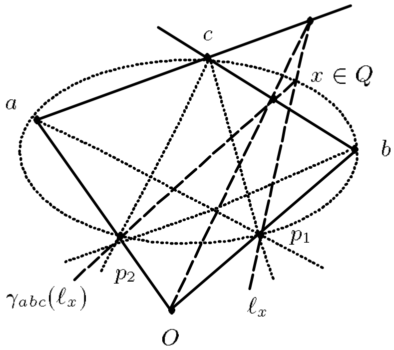

fig. 6.4. Elliptic isomorphism of pencils.

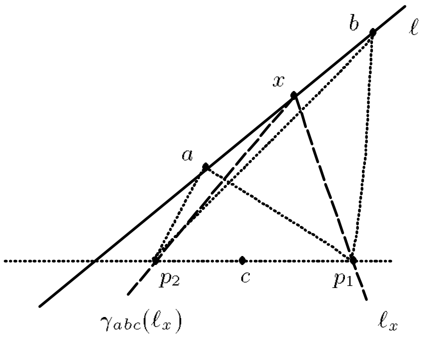

fig. 6.5. Parabolic isomorphism of pencils.

(B) Parabolic case: Q is reducible, i. e. splits in two lines: $(p_1 p_2)$ and, say, $\ell = (a\,b)$. This happens when $c \in (p_1 p_2)$ (recall that no 2 points from a, b, c are collinear with any of p_i). In this case the incidence graph for γ_{abc} coincides with the line ℓ (see fig. 6.5).

Dualizing these examples, we get the geometrical classification of projective linear isomorphisms $\ell_1 \xrightarrow{\sim} \ell_2$ between two given lines on $\mathbb{P}_2$.

Corollary 6.4 *There are exactly two types of projective linear isomorphisms $\ell_1 \xrightarrow{\sim} \ell_2$. Elliptic isomorphisms γ_Q correspond bijectively to the non singular conics Q touching both ℓ_1 and ℓ_2. Such γ_Q sends $x \longmapsto y$ iff the line $(x\,y)$ is tangent to Q (see fig 6.6). Parabolic isomorphisms γ_L are parametrized by the points $L \in \mathbb{P}_2 \setminus (\ell_1 \cup \ell_2)$. Such γ_L sends $x \longmapsto y$ iff the line $(x\,y)$ pass through L (see fig 6.7).* $\square$

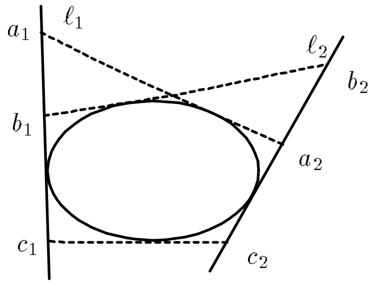

fig. 6.6. Elliptic isomorphism of lines.

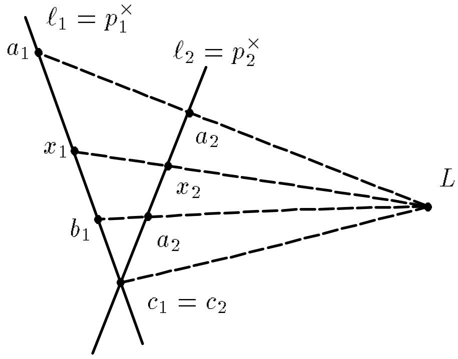

fig. 6.7. Parabolic isomorphism of lines.

<u>MAKING A PROJECTIVE ISOMORPHISM BY THE RULER.</u> If an isomorphism $\ell_1 \xrightarrow{\gamma} \ell_2$ is given by its action on some 3 points, say: $a_1 \longmapsto a_2$, $b_1 \longmapsto b_2$, $c_1 \longmapsto c_2$, then we can find the image $\gamma(x)$ of any $x \in \ell_1$ by the ruler. This is trivial in the parabolic case (see fig 6.7). In the elliptic case the drawing is projectively dual to the one discussed in Example 6.3 (A). Namely (compare fig. 6.4. and fig. 6.8): draw the line $O^\times = (b_1 a_2)$; then pass the line through x and $(a_1 a_2) \cap (c_1 c_2)$ and mark its intersection point with $O^\times$; now $\gamma(x)$ is the intersection of ℓ_2 with the line passing through the last marked point and $(b_1 b_2) \cap (c_1 c_2)$.

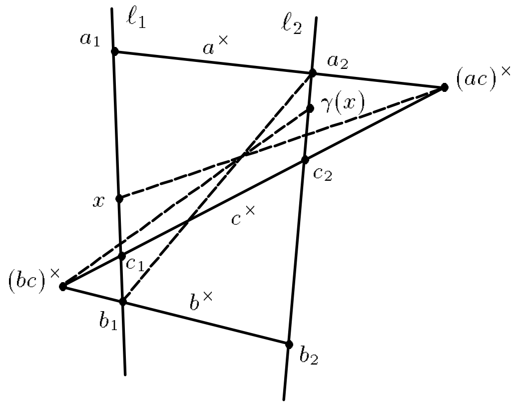

fig. 6.8. Finding $\gamma_Q(x)$.

[4]i. e. a curve traced by the intersection points $\ell \cap \gamma_{abc}(\ell)$ while ℓ runs through $p_1^\times$