Quadrics: linear subspaces laying on a quadric.

COMPLEX PROJECTIVE QUADRICS ON $\mathbb{P}_3$ up to an isomorphism are: *a double plane* $x_0^2 = 0$; *a reducible quadric* $x_0^2 + x_1^2 = 0$, which is a pair of crossing planes; *a simple cone* $x_0^2 + x_1^2 + x_2^2 = 0$, which is a cone with one point vertex over a non singular plane conic; and a non singular quadric $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$. The projective space of all quadrics on $\mathbb{P}_3$ has dimension $3 \cdot 6/2 = 9$. In particular *any 9 points in* $\mathbb{P}_3$ *lay on some quadric.*

Remark 7.1 *Any 3 lines on* $\mathbb{P}_3$ *lay on some quadric.*

Proof. Take 3 distinct points on each line and draw a quadric through these 9 points. □

Note that if 3 lines are pairwise skew, then a quadric containing these lines must be non singular. If 2 of the lines are skew, then such quadric is ether reducible or non singular (but not the cone or the double plane).

SEGRE QUADRIC IN $\mathbb{P}_3$. Let U_-, U_+ be 2-dimensional spaces and $W = Hom(U_-, U_+)$ be the space of all linear maps. Then $\mathbb{P}_3 = \mathbb{P}(W)$ consists of non zero linear maps considered up to proportionality. The degenerate maps form a quadric $Q_S \subset \mathbb{P}_3$, which is called *a Segre quadric*. In the matrix notations, Q_S is presented by the determinantal equation

$$\det \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} = \alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10} = 0 \ . \tag{3}$$

This quadric is bijectively parametrized by the points of $\mathbb{P}_1 \times \mathbb{P}_1 = \mathbb{P}(U_-^*) \times \mathbb{P}(U_+)$ as follows. *A Segre embedding*

$$\mathbb{P}(U_-^*) \times \mathbb{P}(U_+) \xhookrightarrow{s} \mathbb{P}(Hom(U_-, U_+))$$

sends a pair $(\xi, v) \in U_-^* \times U_+$ to the degenerate operator $U_- \xrightarrow{\xi \otimes v} U_+$ which is defined by the rule $u \longmapsto \xi(u)\, v \in U_+$ for all $u \in U_-$. So, the kernel $\ker(\xi \otimes v)$ coincides with the zero subspace of the linear form ξ and the image $\mathrm{im}\,(\xi \otimes v)$ is spaned by v. Since any non zero degenerate operator $U_- \xrightarrow{\varphi} U_+$ has the 1-dimensional kernel and the 1-dimensional image, it may be written as $\varphi = \xi \otimes v$ for some $\xi \in U_-^*$, $v \in U_+$, which are well defined by φ up to proportionality. So, the Segre embedding maps $\mathbb{P}_1 \times \mathbb{P}_1$ bijectively onto Q_S.

In terms of coordinates, if $\xi = (\xi_0 : \xi_1) \in U_-^*$ and $v = (t_0 : t_1) \in U_+$, then the operator $\xi \otimes v$ sends the first basic vector $(1, 0) \in U_-$ to $\xi_0 \cdot v$ and the second basic vector $(0, 1) \in U_-$ to $\xi_1 \cdot v$. Hence, $\xi \otimes v$ has a the matrix $\begin{pmatrix} \xi_0 t_0 & \xi_1 t_0 \\ \xi_0 t_1 & \xi_1 t_1 \end{pmatrix}$. So, the Segre embedding gives a parametrization for the quadric (3) by the formulas

$$\alpha_{00} = \xi_0 t_0 \ , \quad \alpha_{01} = \xi_1 t_0 \ , \quad \alpha_{10} = \xi_0 t_1 \ , \quad \alpha_{11} = \xi_1 t_1 \ .$$

where $\big((\xi_0 : \xi_1) \, , \, (t_0 : t_1) \big) \in \mathbb{P}_1 \times \mathbb{P}_1$. In other words, any rank one 2×2 - matrix (a_{ij}) has proportional rows and proportional columns and their ratios

$$([\text{row } 1)] : [\text{row } 2]) \stackrel{\text{def}}{=} (t_0 : t_1) \qquad \text{and} \qquad ([\text{col } 1)] : [\text{col } 2]) \stackrel{\text{def}}{=} (\xi_0 : \xi_1) \ .$$

define the matrix up to a scalar factor.

TWO RULINGS FOR NON SINGULAR QUADRIC ON $\mathbb{P}_3$. Note that $\mathbb{P}_1 \times \mathbb{P}_1$ is ruled by two families of “coordinate” lines $\xi \times \mathbb{P}_1$ and $\mathbb{P}_1 \times t$. They are called *a first* and *a second* line families.

Lemma 7.2 *The Segre embedding sends each coordinate line family to the ruling of* Q_S *by a family of pairwise skew lines. These two line families exhaust all the lines on* Q_S *and any two lines from different families are intersecting.*

Proof. The line $\xi \times \mathbb{P}_1$, where $\xi = (\xi_0 : \xi_1) \in \mathbb{P}(U_-^*)$, goes to the set of all degenerated matrices with the fixed column ratio, i. e. to the line given in $\mathbb{P}_3$ by two linear equations $a_{00} : a_{01} = a_{10} : a_{11} = \xi_0 : \xi_1$ Analogously, $s(\mathbb{P}_1 \times t)$, where $t = (t_0 : t_1) \in \mathbb{P}(U_+)$, goes to the line given in $\mathbb{P}_3$ by $a_{00} : a_{10} = a_{01} : a_{11} = t_0 : t_1$. Since the Segre embedding is bijective, each line family consists of pairwise skew lines, any two lines from the different families are intersecting, and for any $x \in Q_S$ there is a pair of the different type lines passing through x. So, $Q_S \cap T_x Q_S$ is the reducible conic, which splits into the last pair of lines. In particular, there are no other lines on Q_S. □

Corollary 7.3 *There exist a unique quadric passing through 3 given pairwise skew lines in $\mathbb{P}_3$. It is ruled by all lines intersecting each of 3 given lines.*

Proof. Let a quadric Q contain 3 given lines. Since Q is non singular, it can be ruled by two line families via Segre. 3 given lines must belong to the same, say, to the first family. Any line from the second family intersect all the lines of the first family (in particular, 3 given lines). Moreover, any line on $\mathbb{P}_3$ which intersect 3 given lines must lay on Q and belong to the second family. $\square$

Exercise 7.4. How many lines intersect 4 given pairwise skew lines in $\mathbb{P}_3$?

HINT. Draw all lines intersecting some 3 of given 4 and look at the interaction of this figure with the remaining 4-th line.

Exercise 7.5*. Do the previous exercise for $\mathbb{A}^3$ instead of $\mathbb{P}_3$ and also over $\mathbb{R}$ instead of $\mathbb{C}$. There will be several answers depending on the line configuration. Find all possible answers and indicate which of them are stable under small perturbations.

LINEAR SUBSPACES LAYING ON A NON SINGULAR QUADRIC . Lemma 7.2 can be extended in higher dimensions. Let $Q_n \subset \mathbb{P}_n = \mathbb{P}(V)$ be non singular quadric and $L = \mathbb{P}(W) \subset Q_n$.

Theorem 7.6 $\dim L \leqslant \left[\frac{n-1}{2}\right]$, *where* $[\,*\,]$ *means the integer part.*

Proof. Let Q_n be given by a quadratic form q with the polarization $\widetilde{q}$. Then

$$L \subset Q_n \iff \widetilde{q}(w_1, w_2) = 0 \;\forall\, w_1, w_2 \in W \iff \widehat{q}(W) \subset \operatorname{Ann}(W) = \{\, \xi \in V^* \mid \xi(w) = 0 \quad \forall\, w \in W \,\},$$

where $\widehat{q} : v \longmapsto \widetilde{q}(v, *)$ is the correlation associated with Q_n. This correlation is injective, because of Q_n is non singular. So, $\dim W \leqslant \dim \operatorname{Ann} W = \dim V - \dim W$. Hence, $\dim L = \dim W - 1 \leqslant (\dim V)/2 - 1 = [(n-1)/2]$. $\square$

Lemma 7.7 *cork* $(H \cap Q_n) \leqslant 1$ *for any codimension 1 hyperplane* $H \subset \mathbb{P}_n$.

Proof. If $H = \mathbb{P}(W)$, then $\dim \ker (q|_W) \leqslant \dim\big(W \cap \widehat{q}^{-1}(\operatorname{Ann} W) \big) \leqslant \dim \widehat{q}^{-1}(\operatorname{Ann} W) = \dim \operatorname{Ann} W = 1$. $\square$

Lemma 7.8 *For any* $x \in Q_n$, $Q_n \cap T_x Q_n$ *is a simple cone with the vertex* x *over a non singular quadric* Q_{n-2} *in a* $(n-2)$*-dimensional projective subspace in* $T_x Q_n \setminus x$.

Proof. Since $T_x Q = \mathbb{P}(\ker \widetilde{q}(x, *))$ and $\widetilde{q}(x, x) = q(x) = 0$, the restriction of q onto $T_x Q$ has at least 1-dimensional kernel presented by x itself. By the previous lemma this kernel is spaned by x. $\square$

Theorem 7.9 *Let* $d_n = \left[\frac{n-1}{2}\right]$ *be the upper bound from Theorem 7.6. The* d_n*-dimensional subspaces* $L \subset Q_n$ *passing through a given point on* Q_n *are parametrized bijectively by the all* $(d_n - 1)$*-dimensional subspaces laying on* Q_{n-2}.

Proof. Fix some $(n-1)$-dimensional projective subspace $H \subset T_x Q \setminus x$ and present $Q_n \cap T_x Q_n$ as a simple cone ruled by lines passing through x and some $Q_{n-2} \subset H$. Since any $L \subset Q_n$ which pass through x is contained inside $Q_n \cap T_x Q_n$, it has to be the linear span of x and some $(d_n - 1)$-dimensional subspace $L' \subset Q_{n-2}$. $\square$

For example, there are only 0-dimensional subspaces on Q_1 and Q_2. Further, there are no planes on Q_3 and Q_4 but any point of Q_3 belongs to two lines[1] and any point of Q_4 belongs to the 1-dimensional family of lines[2]. A non singular quadric $Q_5 \subset \mathbb{P}_5$ contains at most the planes. For any point $x \in Q_5$ there are two 1-dimensional families of planes passing through x. Each family is parametrized by the corresponding family of lines on $Q_3 \simeq \mathbb{P}_1 \times \mathbb{P}_1$, i. e. by $\mathbb{P}_1$.

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[1] they correspond to the points of Q_1

[2] this family is parametrized by the points of a non singular conic