Tensor Guide.

<u>Multilinear maps.</u> Let $V_1, V_2, \dots, V_n$ and W be vector spaces of dimensions $d_1, d_2, \dots, d_n$ and m over an arbitrary field k with $\operatorname{char} k \neq 2$. A map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W$ is called *multilinear* if in each argument

$$\varphi(\dots, \lambda v' + \mu v'', \dots) = \lambda\, \varphi(\dots, v', \dots) + \mu\, \varphi(\dots, v'', \dots)$$

when all the other remain to be fixed. The multilinear maps $V_1 \times V_2 \times \cdots \times V_n \longrightarrow W$ form a vector space of dimension $m \cdot \prod d_\nu$. Namely, if we fix a base $\{e_1^{(i)}, e_2^{(i)}, \dots, e_{d_i}^{(i)}\}$ for each V_i and a base $\{e_1, e_2, \dots, e_m\}$ for W, then any multilinear map φ is uniquely defined by its values at all combinations of the base vectors:

$$\varphi\big(e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \dots, e_{\alpha_n}^{(n)}\big) = \sum_\nu a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)} \cdot e^\nu \in W$$

As soon as $m \cdot \prod d_\nu$ numbers $a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)} \in k$ are given, the map φ is well defined by the multilinearity. It sends a collection of vectors $(v_1, v_2, \dots, v_n)$, where $v_i = \sum\limits_{\alpha_i=1}^{d_i} x_{\alpha_i}^{(i)} e_{\alpha_i}^{(i)} \in V_i$ for $1 \leqslant i \leqslant n$, to

$$\varphi(v_1, v_2, \dots, v_n) = \sum_{j=1}^{m} \Big(\sum_{\alpha_1, \alpha_2, \dots, \alpha_n} a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)} \cdot x_{\alpha_1}^{(1)} \cdot x_{\alpha_2}^{(2)} \cdot \cdots \cdot x_{\alpha_n}^{(n)} \Big) \cdot e^\nu \in W$$

the numbers $a_\nu^{(\alpha_1, \alpha_2, \dots, \alpha_n)}$ can be considered as elements of some "$(n+1)$-dimensional format matrix of the size $m \times d_1 \times d_2 \times \cdots \times d_n$", if you can imagine such a thing[1].

Exercise 8.1. Check that a collection $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \cdots \times V_n$ doesn't contain zero vector iff there exists a multilinear map φ (to somewhere) such that $\varphi(v_1, v_2, \dots, v_n) \neq 0$.

Exercise 8.2. Check that a multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} U$ followed by *a linear operator* $U \xrightarrow{F} W$ is the multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{F\circ\varphi} W$ as well.

<u>Tensor product of vector spaces.</u> Let $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} U$ be a fixed multilinear map. Then for any vector space W we have a composition operator

$$\begin{pmatrix} \text{the space } Hom(U,W) \text{ of all} \\ \text{linear operators } U \xrightarrow{F} W \end{pmatrix} \xrightarrow{F \longmapsto F\circ\tau} \begin{pmatrix} \text{the space of all multilinear maps} \\ \underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\varphi} W \end{pmatrix} \tag{4}$$

A multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} U$ is called *universal* if the composition operator (4) is an isomorphism for any vector space W. In other words, for any W and any multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W$ there exist a unique linear operator $U \xrightarrow{F} W$ such that $\varphi = F\circ\tau$, i. e. the commutative diagram

$$\begin{tikzcd}
 & U \arrow[dd, dashed, "F"] \\
V_1 \times V_2 \times \cdots \times V_n \arrow[ur, "\tau"] \arrow[dr, "\varphi"'] & \\
 & W
\end{tikzcd}$$

can be always closed by a unique linear dotted row.

<u>Claim 8.3</u> *Let $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau_1} U_1$ и $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau_2} U_2$ be two universal multilinear maps. Then there exists a unique linear isomorphism $U_1 \xrightarrow{\iota} U_2$ such that $\tau_2 = \iota\tau_1$.*

[1] usual $d \times m$ - matrices, which present the *linear* maps $V \longrightarrow W$, have just 2-dimensional format

Proof. Since both U_1, U_2 are universal, there are unique linear operators $U_1 \xrightarrow{F_{21}} U_2$ and $U_2 \xrightarrow{F_{12}} U_1$ mounted in the diagrams

$$\begin{tikzcd}
U_1 \arrow[rr, equal, "\mathrm{Id}_{U_1}"] & & U_1 \arrow[dd, dashed, "F_{21}"] \\
& V_1 \times V_2 \times \cdots \times V_n \arrow[ul, "\tau_1"'] \arrow[ur, "\tau_1"] \arrow[dl, "\tau_2"] \arrow[dr, "\tau_2"'] & \\
U_2 \arrow[uu, dashed, "F_{12}"] \arrow[rr, equal, "\mathrm{Id}_{U_2}"'] & & U_2
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
 & & U_2 \arrow[dl, dotted, "F_{12}"'] \arrow[dd, equal, "\mathrm{Id}_{U_2}"] \\
V_1 \times V_2 \times \cdots \times V_n \arrow[urr, "\tau_2"] \arrow[r, "\tau_1"] \arrow[drr, "\tau_2"'] & U_1 \arrow[dr, dotted, "F_{21}"] & \\
 & & U_2
\end{tikzcd}$$

So, the composition $F_{21}F_{12} = \mathrm{Id}_{U_2}$, because of the uniqueness property in the universality of U_2. Similarly, $F_{12}F_{21} = \mathrm{Id}_{U_1}$. $\square$

Claim 8.4 *Let $\{e_1^{(i)}, e_2^{(i)}, \dots, e_{d_i}^{(i)}\} \in V_i$ be a base (for $1 \leqslant i \leqslant n$). Denote by $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ a $(\prod d_i)$ - dimensional vector space whose base vectors are the symbols*

$$e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \ldots \otimes e_{\alpha_n}^{(n)}, \quad 1 \leqslant \alpha_i \leqslant d_i \tag{5}$$

(all possible formal "tensor products" of base vectors $e_\nu^{(\mu)}$). Then the multilinear map

$$V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} V_1 \otimes V_2 \otimes \cdots \otimes V_n$$

which sends a base vector collection $(e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_n}) \in V_1 \times V_2 \times \cdots \times V_n$ to the corresponding base vector (5) is universal.

Proof. Let $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\varphi} W$ be a multilinear map and $V_1 \otimes V_2 \otimes \cdots \otimes V_n \xrightarrow{F} W$ be a linear operator. Comparing the values at the base vectors, we see that

$$\varphi = F \circ \tau \iff F(\, e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \ldots \otimes e_{\alpha_n}^{(n)}\,) = \varphi(e_{\alpha_1}^{(1)}, e_{\alpha_2}^{(2)}, \ldots, e_{\alpha_n}^{(n)}) \,.$$

$\square$

THE SEGRE EMBEDDING. The vector space $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is called *a tensor product* of $V_1, V_2, \dots, V_n$. The universal multilinear map $V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\tau} V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is called *a tensor multiplication.* For a collection of vectors $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \cdots \times V_n$ the image $\tau(v_1, v_2, \dots, v_n)$ is denoted by $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ and called *a tensor product* of these vectors. All such products are called *decomposable tensors*. Of course, not all the vectors of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ are decomposable and $\operatorname{im} \tau$ isn't a vector subspace in $V_1 \otimes V_2 \otimes \cdots \otimes V_n$, because τ is *multi*linear but not linear. However, the linear span of decomposable tensors exhausts the whole of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$. Geometrically, the tensor multiplication gives a map

$$\mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \cdots \times \mathbb{P}(V_n) \overset{s}{\hookrightarrow} \mathbb{P}(V_1 \otimes V_2 \otimes \cdots \otimes V_n)$$

called *a Segre embedding*. If $d_i = \dim V_i = m_i + 1$, then the Segre embedding is a bijection between $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2} \times \cdots \times \mathbb{P}_{m_n}$ and *a Segre subvariety* which consist of all decomposable tensors considered up to proportionality. The Segre subvariety has dimension $(\sum m_i)$, doesn't belong to any hyperplane, is ruled by n families of linear subspaces, and lives in $\mathbb{P}_N$ with $N = -1 + \prod(1 + m_i)$.

Example 8.5: the Segre embedding $\mathbb{P}_{m_1} \times \mathbb{P}_{m_2} \hookrightarrow \mathbb{P}_{m_1+m_2+m_1m_2}$ sends $x = (x_0 : x_1 : \cdots : x_{m_1}) \in \mathbb{P}_{m_1}$ and $y = (y_0 : y_1 : \cdots : y_{m_2}) \in \mathbb{P}_{m_2}$ to the point $s(x, y) \in \mathbb{P}_{m_1+m_2+m_1m_2}$ whose $(1+m_1)(1+m_2)$ homogeneous coordinates are all possible products $x_i y_j$ with $0 \leqslant i \leqslant m_1$ and $0 \leqslant j \leqslant m_2$. To visualize this thing, take $\mathbb{P}_{m_1} = \mathbb{P}(V^*)$, $\mathbb{P}_{m_2} = \mathbb{P}(W)$, and $\mathbb{P}_{m_1+m_2+m_1m_2} = \mathbb{P}(Hom(V, W))$, where $Hom(V, W)$ is the space of all linear maps. Then the Segre map sends a pair $(\xi, w) \in V^* \times W$ to the linear map $\xi \otimes w$, which acts by the rule $v \longmapsto \xi(v) \cdot w$.

Exercise 8.6. Check that a map $V^* \times W \longrightarrow Hom(V, W)$ which sends (ξ, w) to the operator $v \longmapsto \xi(v) \cdot w$ is *the universal bilinear map* (so, there is a canonical isomorphism $V^* \otimes W \simeq Hom(V, W)$)

Exercise 8.7. Check that for $\xi = (x_0, x_1, \ldots, x_{m_1}) \in V^*$ and $w = (y_0, y_1, \ldots, y_{m_2}) \in W$ operator $\xi \otimes w$ has the matrix $a_{ij} = x_j y_i$.

Since any operator $\xi \otimes w$ has 1-dimensional image, the corresponding matrix has rank 1. On the other side, any rank 1 matrix has proportional columns. Hence, the corresponding operator has 1-dimensional image, say spaned by $w \in W$, and takes $v \longmapsto \xi(v)\, w$, where the coefficient $\xi(v) \in k$ depends on v linearly. So, the image of the Segre embedding consists of all rank 1 operators up to proportionality. In particular, it may be presented as the intersection of quadrics $\det \begin{pmatrix} a_{ij} & a_{ik} \\ a_{\ell j} & a_{\ell k} \end{pmatrix} = a_{ij}a_{\ell k} - a_{ik}a_{\ell j} = 0$ (i. e. by vanishing all 2×2 - minors for the matrix $(a_{\mu\nu})$).

<u>Tensor algebra of a vector space.</u> If $V_1 = V_2 = \cdots = V_n = V$, then $V^{\otimes n} \stackrel{\text{def}}{=} \underbrace{V\otimes V\otimes\cdots\otimes V}_{n}$ is called *an n-th tensor power* of V. All tensor powers are combined in the infinite dimensional non commutative graded algebra $\mathsf{T}^\bullet V = \bigoplus_{n\geqslant 0} V^{\otimes n}$, where $V^{\otimes 0} \stackrel{\text{def}}{=} k$.

Exercise 8.8. Using the universality, show that there are canonical isomorphisms

$$\left(V^{\otimes n_1}\otimes V^{\otimes n_2}\right)\otimes V^{\otimes n_3} \simeq V^{\otimes n_1}\otimes\left(V^{\otimes n_2}\otimes V^{\otimes n_3}\right) \simeq V^{\otimes(n_1+n_2+n_3)}$$

which make the vector's tensoring to be well defined associative multiplication on $\mathsf{T}^\bullet V$.

Algebraically, $\mathsf{T}^\bullet V$ is what is called "a free associative k-algebra generated by V" ([2]). Practically, this means that as soon as a basis $\{e_1, e_2, \dots, e_d\}\subset V$ is fixed, then $\mathsf{T}^\bullet V$ turns into the space of the formal finite linear combinations of words which consist of the letters e_i separated by $\otimes$. The words are multiplied by writing after one other consequently. This multiplication is extended on the linear combinations of the words by the usual distributivity rules.

<u>Duality.</u> The spaces $V^{\otimes n} = \underbrace{V\otimes V\otimes\cdots\otimes V}_{n}$ and $V^{*\otimes n} = \underbrace{V^*\otimes V^*\otimes\cdots\otimes V^*}_{n}$ are canonically dual to each other. The pairing between $v = v_1\otimes v_2\otimes\cdots\otimes v_n\in V^{\otimes n}$ and $\xi = \xi_1\otimes\xi_2\otimes\cdots\otimes\xi_n\in V^{*\otimes n}$ is given by *a full contraction*

$$\langle\, v\,,\,\xi\,\rangle \stackrel{\text{def}}{=} \prod_{i=1}^{n}\xi_i(v_i)\,. \tag{6}$$

Let $\{e_1, e_2, \dots, e_n\}\subset V$ and $\{\xi_1, \xi_2, \dots, \xi_n\}\subset V^*$ be dual bases, i. e. $\xi_i(e_j) = 0$ for $i\neq j$ and $\xi_i(e_i) = 1$. Then the base words $\{\, e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_r}\,\}$ and $\{\,\xi_{j_1}\otimes\xi_{j_2}\otimes\cdots\otimes\xi_{j_s}\,\}$ form the dual bases for $\mathsf{T}^\bullet V$ and $\mathsf{T}^\bullet V^*$ with respect to the full contraction. So, $V^{\otimes n\,*}\simeq V^{*\otimes n}$.

On the other side, the space $\left(V^{\otimes n}\right)^*$ is naturally identified with the space of all multilinear forms $\underbrace{V\times V\times\cdots\times V}_{n}\longrightarrow k$, because $V^{\otimes n}$ is universal. So, there exists a canonical isomorphism between $V^{*\otimes n} = \underbrace{V^*\otimes V^*\otimes\cdots\otimes V^*}_{n}$ and the space of multilinear forms in n arguments from V. It sends a tensor $\xi = \xi_1\otimes\xi_2\otimes\cdots\otimes\xi_n\in V^{*\otimes n}$ to the form $(v_1, v_2, \dots, v_n)\longmapsto\prod_{i=1}^{n}\xi_i(v_i)$.

<u>Partial contractions.</u> Let $p, q\in\mathbb{N}$ and $I = \{i_1, i_2, \dots, i_m\}\subset\{1, 2, \dots, p\}$, $J = \{j_1, j_2, \dots, j_m\}\subset\{1, 2, \dots, q\}$ be two index subsets of the same cardinality m. A linear operator

$$\underbrace{V^*\otimes V^*\otimes\cdots\otimes V^*}_{p}\otimes\underbrace{V\otimes V\otimes\cdots\otimes V}_{q}\xrightarrow{c_{IJ}}\underbrace{V^*\otimes V^*\otimes\cdots\otimes V^*}_{p-m}\otimes\underbrace{V\otimes V\otimes\cdots\otimes V}_{q-m}$$

which sends $\xi_1\otimes\xi_2\otimes\cdots\otimes\xi_p\otimes v_1\otimes v_2\otimes\cdots\otimes v_q$ to $\prod_{\nu=1}^{m}\xi_{i_\nu}(v_{j_\nu})\cdot\bigotimes_{i\notin I}\xi_i\otimes\bigotimes_{j\notin J}v_j$ is called *a partial contraction* in the indexes I and J. It's important that any two contractions in disjoint index sets commute with each other.

<u>Example 8.9:</u> the contraction between a vector and a multilinear form. Consider a multilinear form $\varphi(v_1, v_2, \dots, v_n)$ as the tensor from $V^{*\otimes n}$ and contact it in the first index with a vector $v\in V$. The result belongs to $V^{*\otimes(n-1)}$ and gives a multilinear form in $(n-1)$ arguments. This form is denoted by $i_v\varphi$ and called *an inner product of v and φ*.

Exercise 8.10. Check that $i_v\varphi(w_1, w_2, \dots, w_{n-1}) = \varphi(v,\, w_1, w_2, \dots, w_{n-1})$, i. e. the inner multiplication by v is just the fixation of v in the first argument.

<u>Claim 8.11</u> *For any $v\in V$ the inner multiplication $V^{*\otimes n}\xrightarrow{i_v}V^{*\otimes(n-1)}$ is dual to the left tensoring*

$$V^{\otimes(n-1)}\xrightarrow{\mu_v:\,t\longmapsto v\otimes t}V^{\otimes n}$$

[2] If you like it, make the following formal exercise: deduce from the universality that for any associative k-algebra A and vector space map $V\xrightarrow{f}A$ there exists a unique algebra homomorphism $\mathsf{T}^\bullet V\xrightarrow{\alpha}A$ such that $\alpha|_V = f$

with respect to the full contraction, i. e. $\langle \mu_v(t), \varphi \rangle = \langle t, i_v(\varphi) \rangle \quad \forall t \in V^{\otimes(n-1)} \quad \forall \varphi \in V^{*\otimes n}$.

Proof. Indeed, for any $w_1 \otimes w_2 \otimes \cdots \otimes w_{n-1} \in V^{\otimes(n-1)}$ and $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in V^{*\otimes n}$ we have

$$\begin{aligned}\langle \mu_v(w_1 \otimes w_2 \otimes \cdots \otimes w_{n-1}), \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \rangle &= \langle v \otimes w_1 \otimes w_2 \otimes \cdots \otimes w_{n-1}, \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \rangle = \\ &= \xi_1(v)\,\xi_2(w_1) \cdots \xi_n(w_{n-1}) = \xi_1(v) \cdot \langle w_1 \otimes w_2 \otimes \cdots \otimes w_{n-1}, \xi_2 \otimes \xi_3 \otimes \cdots \otimes \xi_n \rangle = \\ &= \langle w_1 \otimes w_2 \otimes \cdots \otimes w_{n-1}, i_v(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) \rangle\end{aligned}$$

□

Symmetry properties. A multilinear map $\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\varphi} W$ is called *symmetric* if it doesn't change its value under any permutations of the arguments. If the value of φ is stable under the even permutations and changes the sign under the odd ones, then φ is called *skew symmetric*. Since the composition operator (4) preserves the symmetry properties, for the (skew)symmetric φ the composition operator (4) turns into

$$\begin{pmatrix} \text{the space } Hom(U,W) \text{ of all} \\ \text{linear operators } U \xrightarrow{F} W \end{pmatrix} \xrightarrow{F \mapsto F \circ \varphi} \begin{pmatrix} \text{the space of all (skew)symmetric} \\ \text{multilinear maps } \underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\psi} W \end{pmatrix} \tag{7}$$

A (skew)symmetric multilinear map $\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\varphi} U$ is called *universal* if (7) is an isomorphism for any W. In the symmetric case the universal target space is denoted by S^nV and called n-th *symmetric power* of V. In the skew symmetric case it is called n-th *exterior power* of V and denoted by $\Lambda^n V$.

Exercise 8.12. Show that if they exist, then they are unique up to unique isomorphism which commute with the universal maps.

Symmetric algebra $S^\bullet V$ of V is a factor algebra of the free associative algebra $\mathsf{T}^\bullet V$ by a commutation relations $vw = wv$. More precisely, denote by $\mathcal{I}_{\text{sym}} \subset \mathsf{T}^\bullet V$ a linear span of all tensors

$$\cdots \otimes v \otimes w \otimes \cdots \quad - \quad \cdots \otimes w \otimes v \otimes \cdots,$$

where the both terms are decomposable, have the same degree, and differ only in order of v, w. Clearly, $\mathcal{I}_{\text{sym}}$ is a double-sided ideal in $\mathsf{T}^\bullet V$ generated by a linear span of all the differences $v \otimes w - w \otimes v \in V \otimes V$. The factor algebra

$$S^\bullet V \stackrel{\text{def}}{=} \mathsf{T}^\bullet V / \mathcal{I}_{\text{sym}}$$

is called *a symmetric algebra* of the vector space V. By the construction, it is commutative[3]. Since $\mathcal{I}_{\text{sym}} = \bigoplus_{n \geqslant 0} (\mathcal{I}_{\text{sym}} \cap V^{\otimes n})$ is the direct sum of its homogeneous components, the symmetric algebra is graded:

$$S^\bullet V = \bigoplus_{n \geqslant 0} S^n V \text{ , where } \quad S^n V \stackrel{\text{def}}{=} V^{\otimes n} / (\mathcal{I}_{\text{sym}} \cap V^{\otimes n}) \,.$$

Claim 8.13 *The tensor multiplication followed by the factorization map:*

$$\underbrace{V \times V \times \cdots \times V}_{n} \overset{\tau}{\dashrightarrow} V^{\otimes n} \overset{\pi}{\twoheadrightarrow} S^n(V) \tag{8}$$

gives the universal symmetric multilinear map.

Proof. *Any* multilinear map $V \times V \times \cdots \times V \xrightarrow{\varphi} W$ is uniquely decomposed as $\varphi = F \circ \tau$, where $V^{\otimes n} \xrightarrow{F} W$ is linear. F is factored through π iff $F(\cdots \otimes v \otimes w \otimes \cdots) = F(\cdots \otimes w \otimes v \otimes \cdots)$, i. e. iff $\varphi(\ldots, v, w, \ldots) = \varphi(\ldots, w, v, \ldots)$
□

The graded components S^nV are called *symmetric powers* of V and the map (8) is called *a symmetric multiplication*. If a base $\{e_1, e_2, \ldots, e_d\} \subset V$ is fixed, then S^nV is naturally identified with the space of all homogeneous polynomials of degree n in e_i. Namely, consider the polynomial ring $k[e_1, \ldots, e_d]$ (whose

[3] Again, if you like it, prove that for any commutative k-algebra A and a vector space map $V \xrightarrow{f} A$ there exists a unique homomorphism of commutative algebras $S^\bullet V \xrightarrow{\alpha} A$ such that $\alpha|_V = f$

"variables" are the base vectors e_i) and identify V with the space of all linear homogeneous polynomials in e_i.

Exercise 8.14. Check that the multiplication map

$$\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{(\ell_1,\ell_2,\dots,\ell_n)\longmapsto \prod_{\nu=1}^{n} \ell_\nu} \begin{pmatrix} \text{the homogeneous polynomials} \\ \text{of degree } n \text{ in } e_i \end{pmatrix}$$

is universal.

In particular, $\dim S^nV = \binom{d+n-1}{n}$.

<u>EXTERIOR ALGEBRA $\Lambda^\bullet V$ OF V</u> is a factor algebra of the free associative algebra $\mathsf{T}^\bullet V$ by a skew commutation relations $vw = -wv$. More precisely, consider a double-sided ideal $\mathcal{I}_{\text{skew}} \subset \mathsf{T}^\bullet V$ generated by all sums $v \otimes w + w \otimes v \in V \otimes V$ and put $\Lambda^\bullet V \stackrel{\text{def}}{=} \mathsf{T}^\bullet V/\mathcal{I}_{\text{skew}}$. Exactly as in the symmetric case, the ideal $\mathcal{I}_{\text{skew}}$ is *homogeneous*: $\mathcal{I}_{\text{skew}} = \bigoplus_{n\geqslant 0} (\mathcal{I}_{\text{skew}} \cap V^{\otimes n})$, where $(\mathcal{I}_{\text{skew}} \cap V^{\otimes n})$ is the linear span of all sums

$$\cdots \otimes v \otimes w \otimes \cdots \quad + \quad \cdots \otimes v \otimes w \otimes \cdots$$

(the both items have degree n and differ only in the order of v, w). So, the factor algebra $\Lambda^\bullet V$ is graded by the subspaces $\Lambda^n V \stackrel{\text{def}}{=} V^{\otimes n}/(\mathcal{I}_{\text{skew}} \cap V^{\otimes n})$.

Exercise 8.15. Prove that the tensor multiplication followed by the factorization

$$\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\tau} V^{\otimes n} \xrightarrow{\pi} \Lambda^n(V) \tag{9}$$

gives the universal skew symmetric multilinear map.

The map (9) is called *an exterior* or *skew* multiplication. The skew product of vectors $(v_1, v_2, \dots, v_n)$ is denoted by $v_1 \wedge v_2 \wedge \cdots \wedge v_n$. By the construction, it changes the sign under the transposition of any two consequent terms. So, under any permutation of terms the skew product is multiplied by the sign of the permutation.