Grassmannian polynomials: examples.

GRASSMANNIAN POLYNOMIALS. Let $\{e_1, e_2, \dots, e_d\} \subset V$ be a basis. Then the exterior algebra $\Lambda^\bullet V$ is identified with *a grassmannian polynomial ring* $k\langle e_1, \dots, e_d\rangle$ whose "variables" are the base vectors e_i which *skew commute*, that is, $e_i \wedge e_j = -e_j \wedge e_i$ for all i, j. More precisely, since $V^{\otimes n}$ is spaned by the degree n tensor products $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$, the exterior power $\Lambda^n V$ is linearly spaned by the grassmannian monomials $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$. It follows from skew commutativity that $e_i \wedge e_i = 0$ for all i, that is, a grassmannian monomial vanishes as soon as it becomes of degree more then 1 in some e_i. So, any grassmannian monomial has a unique representation $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$ with $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant d$.

Claim 9.1 *The monomials* $e_I \stackrel{\text{def}}{=} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$, *where* $I = (i_1, i_2, \dots, i_n)$ *runs through the increasing* n*-element subsets in* $\{1, 2, \dots, d\}$, *form a base for* $\Lambda^n V$. *In particular,* $\Lambda^n V = 0$ *for* $n > \dim V$, $\dim \Lambda^n V = \binom{d}{n}$, *and* $\dim k\langle e_1, \dots, e_d\rangle = 2^d$.

Proof. Consider $\binom{d}{n}$-dimensional vector space U whose base consists of the symbols ξ_I, where $I = (i_1, i_2, \dots, i_n)$ runs through the increasing n-element subsets in $\{1, 2, \dots, d\}$. Define a skew symmetric multilinear map

$$V_1 \times V_2 \times \cdots \times V_n \xrightarrow{\alpha} U \ : \ (e_{j_1}, e_{j_2}, \dots, e_{j_n}) \longmapsto \operatorname{sgn}(\sigma) \cdot \xi_I \ ,$$

where $I = (j_{\sigma(1)}, j_{\sigma(2)}, \dots, j_{\sigma(n)},)$ is an *increasing* collection obtained from $(j_1, j_2, \dots, j_n)$ by a (unique) permutation σ. This map is universal. Indeed, for any skew symmetric multilinear map $\underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\varphi} W$ there exists at most one linear operator $U \xrightarrow{F} W$ such that $\varphi = F \circ \alpha$, because it has to act on the base as $F(\xi_I) = \varphi(e_{i_1}, e_{i_2}, \dots, e_{i_n}))$ for all increasing $I = (i_1, i_2, \dots, i_n)$. On the other side, such F really decomposes φ, because $F(\alpha(e_{j_1}, e_{j_2}, \dots, e_{j_n})) = \varphi(e_{j_1}, e_{j_2}, \dots, e_{j_n}))$ for all *not increasing* base collections $(e_{j_1}, e_{j_2}, \dots, e_{j_n}) \subset \underbrace{V \times V \times \cdots \times V}_{n}$ as well. By the universality, there exists a canonical isomorphism between U and $\Lambda^n V$ which sends ξ_I to $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} = e_I$. $\square$

Exercise 9.2. Check that $f(e) \wedge g(e) = (-1)^{\deg(f)\cdot\deg(g)}\, g(e) \wedge f(e)$ for all homogeneous $f(e), g(e) \in k\langle e_1, \dots, e_d\rangle$. In particular, each *even degree* homogeneous polynomial commutes with any grassmannian polynomial.

Exercise 9.3. Describe the center of $k\langle e_1, \dots, e_d\rangle$, i. e. all grassmannian polynomials which commute with everything.

LINEAR BASE CHANGE. Under the linear substitution $e_i = \sum\limits_{j=1}^{d} a_{ij}\, \xi_j$ the base monomials e_I are changed by the new base monomials ξ_I as follows:

$$\begin{aligned} e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} &= \Big(\sum_{j_1} a_{i_1 j_1}\, \xi_j\Big) \wedge \Big(\sum_{j_2} a_{i_2 j_2}\, \xi_j\Big) \wedge \cdots \wedge \Big(\sum_{j_n} a_{i_n j_n}\, \xi_j\Big) = \\ &= \sum_{1 \leqslant j_1 < j_2 < \cdots < j_n \leqslant n}\ \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\, a_{i_1 j_{\sigma(1)}} a_{i_2 j_{\sigma(2)}} \cdots a_{i_n j_{\sigma(n)}}\, \xi_{j_1} \wedge \xi_{j_2} \wedge \cdots \wedge \xi_{j_n} = \sum_J a_{IJ}\, \xi_J \ , \end{aligned}$$

where a_{IJ} is $(n \times n)$-minor of (a_{ij}) placed at $(i_1, i_2, \dots, i_n)$ rows and $(j_1, j_2, \dots, j_n)$ columns, and J runs through all increasing index collections of *the length* $\#J = n$.

Exercise 9.4. Let $|I| = \sum\limits_{\nu} i_\nu$ denote *a weight* of the increasing index collection $I = (i_1, i_2, \dots, i_n)$ of length $\#I = n$. Check that

$$e_I \wedge e_{\widehat{I}} = (-1)^{|I| + \frac{1}{2}\#I(1+\#I)} \cdot e_1 \wedge e_2 \wedge \cdots \wedge e_d \tag{10}$$

for any two *complementary* index collections I and $\widehat{I} \stackrel{\text{def}}{=} \{1, 2, \dots, n\} \setminus I$.

LAPLACE RELATIONS. Let us take two complementary index collections I and $\widehat{I} \stackrel{\text{def}}{=} \{1, 2, \dots, n\} \setminus I$ and do a base change $e_i = \sum_{j=1}^{d} a_{ij}\,\xi_j$ in the identity (10). Then in the left side of (10) we get

$$e_I \wedge e_{\widehat{I}} = \Big(\sum_{\substack{K:\\ \#K=\#I}} a_{IK}\,\xi_K\Big) \wedge \Big(\sum_{\substack{L:\\ \#L=(d-\#I)}} a_{L\widehat{I}}\,\xi_L\Big) =$$

$$= (-1)^{\frac{1}{2}\#I(1+\#I)} \sum_{\substack{K:\\ \#K=\#I}} (-1)^{|K|} a_{IK} a_{\widehat{I}\widehat{K}}\, \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_d\ ,$$

where $\widehat{K} = \{1, 2, \ldots, d\} \setminus K$. The right side of (10) gives $(-1)^{\frac{1}{2}\#I(1+\#I)}(-1)^{|I|}\det(a_{ij})\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_d$. So, for any any collection I of rows in any square matrix (a_{ij}) the following relation holds:

$$\sum_{\substack{K:\\ \#K=\#I}} (-1)^{|K|+|I|} a_{IK} \widehat{a}_{IK} = \det(a_{ij})\ , \tag{11}$$

where $\widehat{a}_{IK} \stackrel{\text{def}}{=} a_{\widehat{I}\widehat{K}}$ denotes the $(d-n)\times(d-n)$ - minor which is complementary[1] to a_{IK} and the summation runs over all $(n \times n)$ - minors a_{IK} contained in the rows $(i_1, i_2, \ldots, i_n)$.

If we take $I \cap J \neq \varnothing$, then starting from $e_I \wedge e_J = 0$ instead of (10) we get by the same calculation the relation

$$\sum_{\substack{K:\\ \#K=\#I}} (-1)^{|K|+|I|} a_{IK} \widehat{a}_{JK} = 0\ , \tag{12}$$

The identities (11) and (12) are known as *Laplace relations*. Let us fix, say lexicografical, order on the set of indices I and arrange all $(n \times n)$-minors a_{IJ} as $\binom{d}{n} \times \binom{d}{n}$ - matrix $A^{(n)} \stackrel{\text{def}}{=} (a_{IJ})$. If we denote by $\widehat{A}^{(n)}$ a matrix whose (IJ)-entry equals $\big((-1)^{|I|+|J|}\widehat{a}_{JI}\big)$, then all the Laplace relations are expressed by the single matrix equality $A \cdot \widehat{A} = \det(a_{ij}) \cdot E$.

<u>GRASSMANNIAN QUADRATIC FORMS.</u> Any homogeneous grassmannian polynomial of degree 2 can be written as

$$\xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4 + \cdots + \xi_{r-1} \wedge \xi_r \tag{13}$$

in some base (over any field k). Namely, we can suppose[2] that our grassmannian quadratic form is

$$q(e) = e_1 \wedge (\alpha_2 e_2 + \cdots + \alpha_n e_n) + e_2 \wedge (\beta_3 e_3 + \cdots + \beta_n e_n) + (\text{terms without } e_1 \text{ and } e_2)$$

where $\alpha_2 \neq 0$ and $\xi_2 \stackrel{\text{def}}{=} \alpha_2 e_2 + \cdots + \alpha_n e_n$ is neither zero not proportional e_1. Taking a new base $\{\zeta_1, \xi_2, \zeta_3, \ldots, \zeta_n\}$, where $\zeta_i = e_i$ for $i \neq 2$, we have to substitute in q the values $e_i = \zeta_i$ except for $e_2 = \alpha_2^{-1}(\xi_2 - \beta_3\zeta_3 - \cdots - \beta_n e_n)$. This leads to

$$q(\zeta) = \zeta_1 \wedge \xi_2 + \xi_2 \wedge (\gamma_3\zeta_3 + \cdots + \gamma_n\zeta_n) + (\text{terms without } \zeta_1 \text{ and } \xi_2)$$

If we take the next new base $\{\xi_1, \xi_2, \zeta_3, \ldots, \zeta_n\}$ with $\xi_1 = e_1 - \gamma_3\zeta_3 - \cdots - \gamma_n\zeta_n$, then q turns into

$$q(\zeta) = \xi_1 \wedge \xi_2 + (\text{terms without } \xi_1 \text{ and } \xi_2)$$

and the procedure can be repeated for the remaining terms.

Exercise 9.5. Let $A = (a_{ij})$ be a skew symmetric matrix (i. e. $a_{ij} = -a_{ji}$) and $q(e) = \sum_{ij} a_{ij}\, e_i \wedge e_j$ be a grassmannian quadratic form. Show that in the representation (13) the number r doesn't depend on the base choice and equals $\operatorname{rk} A$. (In particular, $\operatorname{rk} A$ is always even.)

HINT. Consider a map $V \xrightarrow{x \mapsto x \cdot A} V^*$ given by the matrix $A = (a_{ij})$. This map doesn't depend on the base choice (why?) and its kernel has the dimension $\operatorname{cork} A$ ...

[1] i. e. sitting in the complementary rows and columns

[2] may be after appropriate renumbering of the base vectors