Report on "Young Russian Mathematics" award

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2022 year

Contents

1 Results

1.1 A fully non-abelian analog for the fourth Painlevé equation

In the matrix setting, it turns out that the fourth Painlevé equation has three different analogs labeled as P_4^0, P_4^1, P_4^2 [Bobrova and Sokolov, 2022c]. They were obtained by using a matrix generalization of the Painlevé-Kovalevskaya test, suggested in [Balandin and Sokolov, 1998]. P_4^0 can be written only as a system of first-order ODEs, while the remaining systems are reduced to a second-order ODE. All this systems have arbitrary matrix constants and are integrable in the sense of the zero-curvature condition [Bobrova and Sokolov, 2022c].

In a fully non-abelian setting (non-formally speaking, "the independent variable" is non-abelian), there exists only one analog of the fourth Painlevé equation [Bobrova et al., 2022] that generalizes the quantum fourth Painlevé equation and the matrix P_4^0 system. This analog can be written as a system and was derived by using the solutions of the infinite non-commutative one dimensional Toda lattice. Namely, the solutions of the fully non-commutative fourth Painlevé system can be expressed via the solutions of the infinite non-commutative Toda system. The latter is integrable since it can be solved by Hankel quasideterminants. Besides this property, the fully non-abelian analog is integrable in the sense of the zero-curvature condition.

1.2 Some classes of non-abelian Painlevé type systems

All Painlevé equations are Hamiltonian and can be written as a system of first-order ODEs [Okamoto, 1980]. This systems are non-autonomous and, therefore, the Hamiltonians are not an integral of motions. Since well-known properties of integrable autonomous systems can be naturally generalized to the non-commutative case, it is convenient to consider auxiliary autonomous systems related to the Painlevé systems. Using this approach, we construct two classes of non-abelian Painlevé type systems that are closed under the limiting transitions.

The first class is a class of Hamiltonian non-abelian Painlevé systems [Bobrova and Sokolov, 2022a] that are a slight generalization of the matrix Painlevé systems obtained in [Kawakami, 2015]. To find such a class, we follow the concepts of the Hamiltonian non-abelian systems suggested in [Kontsevich, 1993]. It turns out that systems of Painlevé-5 – Painlevé-1 type have an inessential constant that can be replaced with a non-abelian. To prove the integrability of these systems thus obtained, we find isomonodromic Lax pairs for them.

The second class is integrable in the sense of the existence of a non-abelian first integral that generalizes the Okamoto Hamiltonian [Bobrova and Sokolov, 2022b]. Since the first integrals are not elements of the quotient of a free algebra by its commutant, this class does not contain the Hamiltonian systems. Besides this fact, several known examples appear in this list. We remark that for all of these systems the isomonodromic Lax pairs are presented.

2 Papers

2.1 Published

1. Bobrova I., Retakh V., Rubtsov V., Sharygin G. *A fully noncommutative analog of the Painlevé IV equation and a structure of its solutions*, Journal of Physics A: Mathematical and Theoretical, 55(47): 475205, arXiv:2205.05107

2.2 Submitted

1. Bobrova I., Sokolov V. *Classification of Hamiltonian non-abelian Painlevé type systems*, Journal of Nonlinear Mathematical Physics (in press), arXiv preprint arXiv:2209.00258

2. Bobrova I., Sokolov V. *Non-abelian Painlevé systems with generalized Okamoto integral*, the Contemporary Mathematics, AMS (in press), arXiv preprint arXiv:2206.10580

3 Scientific conferences and schools

3.1 Scientific conferences

1. Ufa Autumn Mathematical School-2022, Zoom, September 28 - October 1
   talk *Non-abelian Painlevé systems with generalized Okamoto integral*

2. Integrable systems and their applications, Sochi, September 12 - 16
   talk *Non-commutative generalisations of the fourth Painlevé equation*

3.2 Schools

1. *EIMI Lie algebras summer school ALGEULER*, Saint-Petersburg

4 Work in scientific centers and international groups

1. Research Assistant, International Laboratory of Cluster Geometry, Faculty of Mathematics, HSE University

2. Invited Researcher, IHES (2 dec 2022 – 28 feb 2023)

5 Teaching

5.1 Courses 2022/2023 (fall)

1. Seminars on Algebra and Analysis, 1st year of Bachelor's programme "Sociology", Faculty of Social Sciences, HSE University

2. Seminars on Calculus, 1st year of Bachelor's programme "Physics", Faculty of Physics, HSE University

5.2 Courses 2021/2022 (spring)

1. Seminars on Probability Theory and Mathematical Statistics, 1st year of Bachelor's programme "Marketing and Market Analytics", Graduate School of Business, HSE University

References

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[Bobrova et al., 2022] Bobrova, I., Retakh, V., Rubtsov, V., and Sharygin, G. (2022). A fully noncommutative analog of the Painlevé IV equation and a structure of its solutions. *Journal of Physics A: Mathematical and Theoretical*, 55(47):475205. arXiv:2205.05107.

[Bobrova and Sokolov, 2022a] Bobrova, I. and Sokolov, V. (2022a). Classification of Hamiltonian non-abelian Painlevé type systems. *arXiv preprint arXiv:2209.00258*.

[Bobrova and Sokolov, 2022b] Bobrova, I. and Sokolov, V. (2022b). Non-abelian Painlevé systems with generalized Okamoto integral. *arXiv preprint arXiv:2206.10580*.

[Bobrova and Sokolov, 2022c] Bobrova, I. and Sokolov, V. (2022c). On matrix Painlevé-4 equations. *Nonlinearity*, 35(12):6528. arXiv:2107.11680, arXiv:2110.12159.

[Kawakami, 2015] Kawakami, H. (2015). Matrix Painlevé systems. *Journal of Mathematical Physics*, 56(3):033503.

[Kontsevich, 1993] Kontsevich, M. (1993). Formal (non)-commutative symplectic geometry, The Gelfand Mathematical Seminars, 1990–1992. *Fields Institute Communications, Birkhäuser Boston*, pages 173–187.

[Okamoto, 1980] Okamoto, K. (1980). Polynomial Hamiltonians associated with Painlevé equations, I. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 56(6):264–268.