Gaussian Multiplicative Chaos and Applications

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Motivations in turbulence

In fluid dynamics, turbulence is a flow regime in which the velocity field presents unsteady vortices on many scales.

Turbulent flows are thus characterized by a highly irregular aspect, an unpredictable behaviour and the existence of many time or space scales. Such flows arise when the source of kinetic energy making the fluid move is much greater than viscosity forces of the fluid. Inversely, the fluid is said to be laminar when it is smooth.

Figure 1: Vortices in a stream

(a) Steam and smould

(b) Wake turbulence

Figure 2: Examples of turbulence

There are many examples of turbulent flows: the mixing of warm and cold air in the atmosphere by wind which causes clear-air turbulence experienced during airplane flight as well as poor astronomical seeing, most of the terrestrial atmospheric circulation, the oceanic and atmospheric mixed layers and intense oceanic currents, the flow conditions in many industrial equipment (such as pipes, ducts, precipitators, gas scrubbers, dynamic scraped surface heat exchangers, etc.) and machines (for instance, internal combustion engines and gas turbines), the external flow over all kind of vehicles such as cars, airplanes, ships and submarines,...

Figure 3: Atmospheric turbulence

Because of the high irregularity of turbulent flows, turbulence problems are always treated statistically rather than deterministically. That is why one can consider turbulence as a part of statistical physics. A specific point of turbulence that is worth being highlighted is the energy cascade: large eddies are unstable and eventually break up originating smaller eddies, and the kinetic energy of the initial large eddy is divided into the smaller eddies that stemmed from it. These smaller eddies undergo the same process, giving rise to even smaller eddies which inherit the energy of their predecessor eddy, and so on. In this way, the energy is passed down from the large scales of the motion to smaller scales until reaching a sufficiently small length scale such that the viscosity of the fluid can effectively dissipate the kinetic energy into internal energy.

From the mathematical angle, energy transfers are understood as follows. It is commonly admitted that the motion of an incompressible flow is ruled by the Navier-Stokes equation:

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u = -\nabla p + \nu \triangle u + f \quad \text{et} \quad \nabla \cdot u = 0. \tag{1}$$

The local dissipation of energy in a set A is given by:

$$\epsilon(A) = \frac{\nu}{2} \int_A \sum_{i,j} (\partial_i u_j + \partial_j u_i)^2 \, dx. \tag{2}$$

In 1941, Kolmogorov suggested a statistical approach of the local dissipation of energy, called the K41 theory. Roughly speaking, he postulated that, for turbulent flows, the local dissipation of energy is

1. *spatially homogeneous*, its distribution is invariant under space translations,
2. *statistically isotrop*, its distribution is invariant under rotations,
3. *self similar*, that is, for some $\alpha > 0$ and for all $\lambda > 0$, we have

   $$\epsilon(\lambda A) \overset{law}{=} \lambda^{\alpha} \epsilon(A).$$

   In that case, the power law spectrum is linear, ie

   $$\mathbb{E}[\epsilon(B(0,r))^q] = r^{\xi(q)}$$

   where ξ is a linear function of q and $B(0,r)$ stands for the ball centered at 0 with radius r.

Largely motivated by the K41 theory, the study of self similar stochastic processes has widely spread out ever since (Brownian motion, fractional Brownian motion, α-stable Levy processes,...).

However, following the celebrated Landau's objection, Kolmogorov and Obukhov revisited in 1962 the K41 theory to postulate what is now known as the KO62 theory. The main change change is the point 3. Experimental facts and datas have shown that the power law spectrum is clearly not linear. The nonlinearity of the spectrum is related with the phenomenon of intermittency in turbulence. Moreover, they both make the assumption of lognormality of the random variable $\epsilon(A)$.

The nonlinearity of the power law spectrum is related to the following notion of stochastic self similarity, called stochastic scale invariance,

$$\epsilon(\lambda A) \overset{loi}{=} \lambda^{\alpha} e^{\Omega_\lambda} \epsilon(A),$$

where $\lambda, \alpha > 0$ and Ω_λ is a random variable independent of $\epsilon(A)$. To understand such a relation, it is convenient to study the simplest situation when the random variable Ω_λ is Gaussian: the underlying theory is then called Gaussian Multiplicative Chaos. The purpose of this lecture is to introduce and study the theory of Gaussian Multiplicative Chaos.

Motivations in finance

The first mathematical model for the evolution of a stock price is due to Bachelier in 1900: it is a Brownian motion. This model had been used for 60 years. In 1965, Samuelson suggested to rather use a geometric Brownian motion. That is what Black, Scholes and Merton successfully did in 1973.

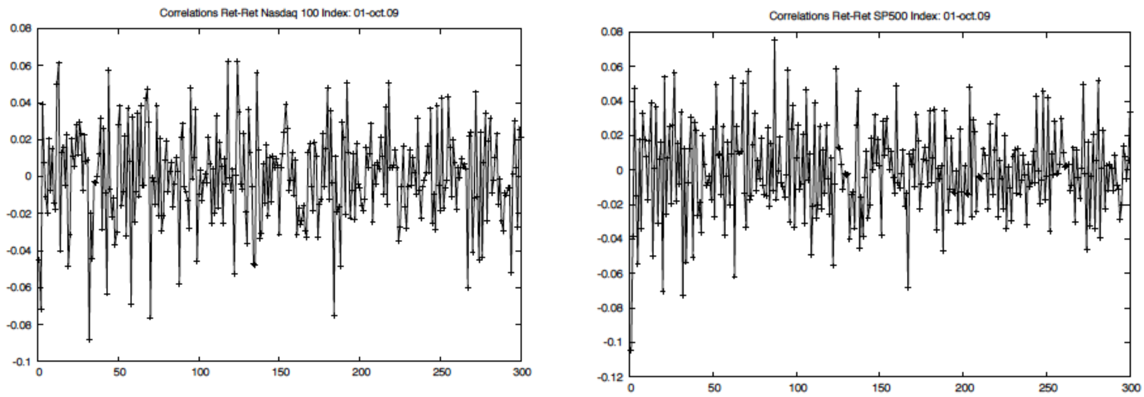
However, the recent market crashes confirmed that this model is far from being suitable. In particular, extreme (or rare) events occur in reality much more often than predicted by such models: with such models, the probability of a crash is so small that the crashes of 1987 and 2009 are quite impossible. Several works, in particular Mandelbrot's and Fama's in the sixties, already pointed out the fact that stock process evolution is far from being Gaussian.

A statistical study of financial markets shows that there are some "universal properties" shared by stock or indices, called "stylized facts" of markets. The log-price X_t of a good model should satisfy:

1. stationarity of th returns, ie $(X_t)_t$ has stationary increments,
2. decorrelation: $\mathbb{E}[X_s(X_t - X_s)] = 0$ for $s < t$ (see figure 4)
3. clong-range correlations of the volatility $< X >$ (see figure 5):

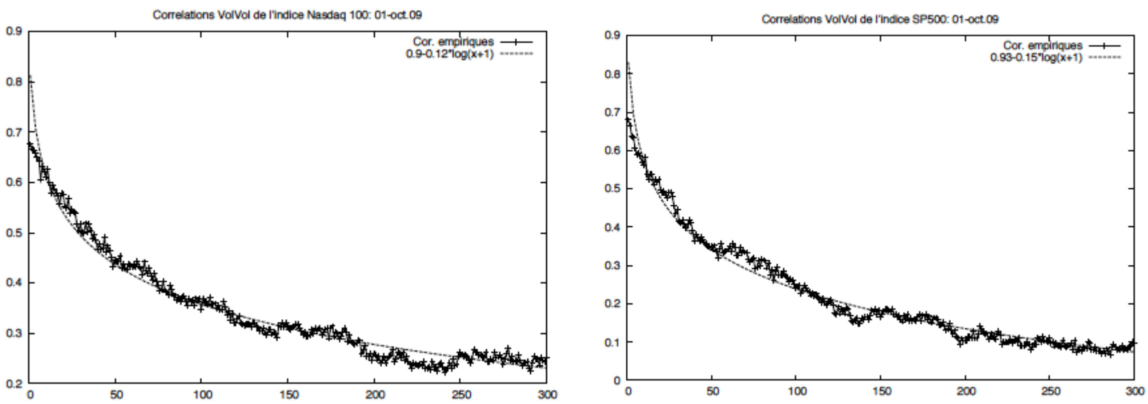
$$\text{Corr}\big(< X >_{0,1}, < X >_{t,t+1} \big) = \frac{A}{(1+t)^{\mu}}$$

with $\mu \in [0; 0.5]$. The limiting case $\mu \to 0$ can also be modelized with correlations of log type: $A - B\ln(1+t)$.



(a) NASDAQ (b) SP500

Figure 4: Daily empirical correlations of indices Nasdaq and SP500 over the period 2001-2009



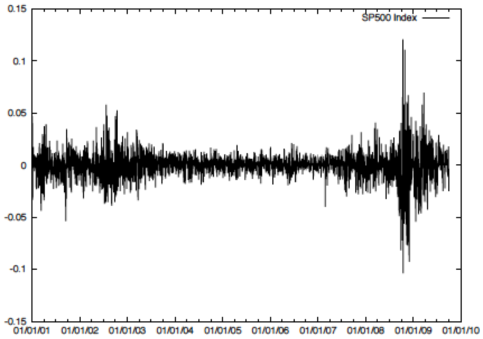
(a) NASDAQ (b) SP500

Figure 5: Empirical correlations of the volatility of indices Nasdaq and SP500 over the period 2001-2009

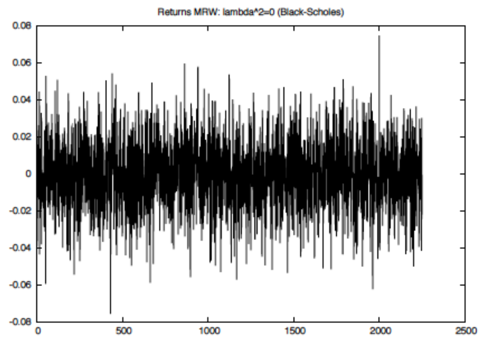
Mandelbrot's idea is to modelize the evolution of the log price with a MRW (Multifractal Random Walk)

$$X_t = B_{M_t},$$

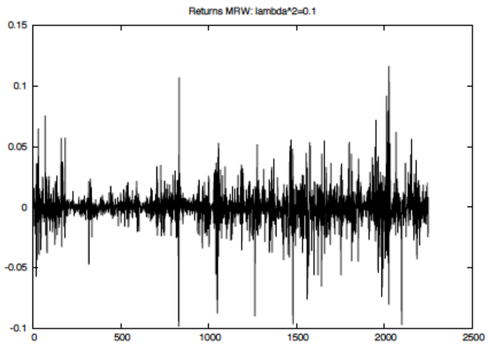
that is a Brownian motion B seen at the time of an increasing stochastic process M with properties very close to those of the local energy dissipation in turbulence, for instance a Gaussian Multiplicative Chaos. The process M can then be seen as the volatility process and possesses intermittency properties that are close to those observed experimentally (see figure 6).



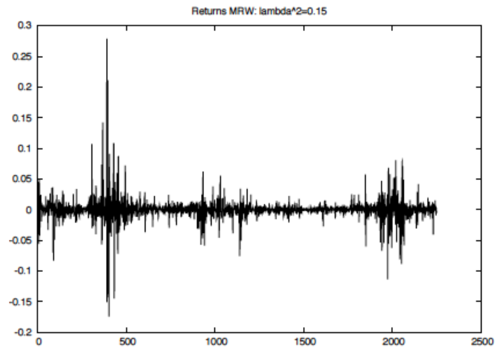
(a) SP500 returns over 2001-2009



(b) Returns simulated with Black-Scholes



(c) Returns simulated with MRW



(d) Returns simulated with MRW

Figure 6: Intermittency in financial markets