Gaussian Multiplicative Chaos and Applications

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Contents

1 Introduction

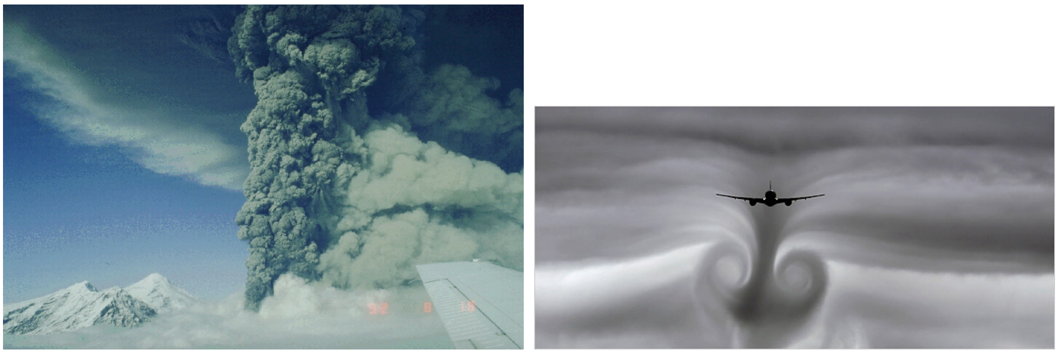
1.1 Motivations in turbulence

In fluid dynamics, turbulence is a flow regime in which the velocity field presents unsteady vortices on many scales.

Turbulent flows are thus characterized by a highly irregular aspect, an unpredictable behaviour and the existence of many time or space scales. Such flows arise when the source of kinetic energy making the fluid move is much greater than viscosity forces of the fluid. Inversely, the fluid is said to be laminar when it is smooth.

Figure 1: Vortices in a stream

There are many examples of turbulent flows: the mixing of warm and cold air in the atmosphere by wind which causes clear-air turbulence experienced during airplane flight as well as poor as-



(a) Steam and smould (b) Wake turbulence

Figure 2: Examples of turbulence

tronomical seeing, most of the terrestrial atmospheric circulation, the oceanic and atmospheric mixed layers and intense oceanic currents, the flow conditions in many industrial equipment (such as pipes, ducts, precipitators, gas scrubbers, dynamic scraped surface heat exchangers, etc.) and machines (for instance, internal combustion engines and gas turbines), the external flow over all kind of vehicles such as cars, airplanes, ships and submarines,...

Figure 3: Atmospheric turbulence

Because of the high irregularity of turbulent flows, turbulence problems are always treated statistically rather than deterministically. That is why one can consider turbulence as a part of statistical physics. A specific point of turbulence that is worth being highlighted is the energy cascade: large eddies are unstable and eventually break up originating smaller eddies, and the kinetic energy of the initial large eddy is divided into the smaller eddies that stemmed from it. These smaller eddies undergo the same process, giving rise to even smaller eddies which inherit the energy of their predecessor eddy, and so on. In this way, the energy is passed down from the large scales of the motion to smaller scales until reaching a sufficiently small length scale such that the viscosity of the fluid can effectively dissipate the kinetic energy into internal energy.

From the mathematical angle, energy transfers are understood as follows. It is commonly admitted that the motion of an incompressible flow is ruled by the Navier-Stokes equation:

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u = -\nabla p + \nu \triangle u + f \quad \text{et} \quad \nabla \cdot u = 0. \tag{1}$$

The local dissipation of energy in a set A is given by:

$$\epsilon(A) = \frac{\nu}{2} \int_A \sum_{i,j} (\partial_i u_j + \partial_j u_i)^2 \, dx. \tag{2}$$

In 1941, Kolmogorov suggested a statistical approach of the local dissipation of energy, called the K41 theory. Roughly speaking, he postulated that, for turbulent flows, the local dissipation of energy is

1. *spatially homogeneous*, its distribution is invariant under space translations,

2. *statistically isotrop*, its distribution is invariant under rotations,

3. *self similar*, that is, for some $\alpha > 0$ and for all $\lambda > 0$, we have

   $$\epsilon(\lambda A) \overset{law}{=} \lambda^\alpha \epsilon(A).$$

   In that case, the power law spectrum is linear, ie

   $$\mathbb{E}[\epsilon(B(0,r))^q] = r^{\xi(q)}$$

   where ξ is a linear function of q and $B(0,r)$ stands for the ball centered at 0 with radius r.

Largely motivated by the K41 theory, the study of self similar stochastic processes has widely spread out ever since (Brownian motion, fractional Brownian motion, α-stable Levy processes,...).

However, following the celebrated Landau's objection, Kolmogorov and Obukhov revisited in 1962 the K41 theory to postulate what is now known as the KO62 theory. The main change change is the point 3. Experimental facts and datas have shown that the power law spectrum is clearly not linear. The nonlinearity of the spectrum is related with the phenomenon of intermittency in turbulence. Moreover, they both make the assumption of lognormality of the random variable $\epsilon(A)$.

The nonlinearity of the power law spectrum is related to the following notion of stochastic self similarity, called stochastic scale invariance,

$$\epsilon(\lambda A) \overset{loi}{=} \lambda^\alpha e^{\Omega_\lambda} \epsilon(A),$$

where $\lambda, \alpha > 0$ and Ω_λ is a random variable independent of $\epsilon(A)$. To understand such a relation, it is convenient to study the simplest situation when the random variable Ω_λ is Gaussian: the underlying theory is then called Gaussian Multiplicative Chaos. The purpose of this lecture is to introduce and study the theory of Gaussian Multiplicative Chaos.

1.2 Motivations in finance

The first mathematical model for the evolution of a stock price is due to Bachelier in 1900: it is a Brownian motion. This model had been used for 60 years. In 1965, Samuelson suggested to rather use a geometric Brownian motion. That is what Black, Scholes and Merton successfully did in 1973.

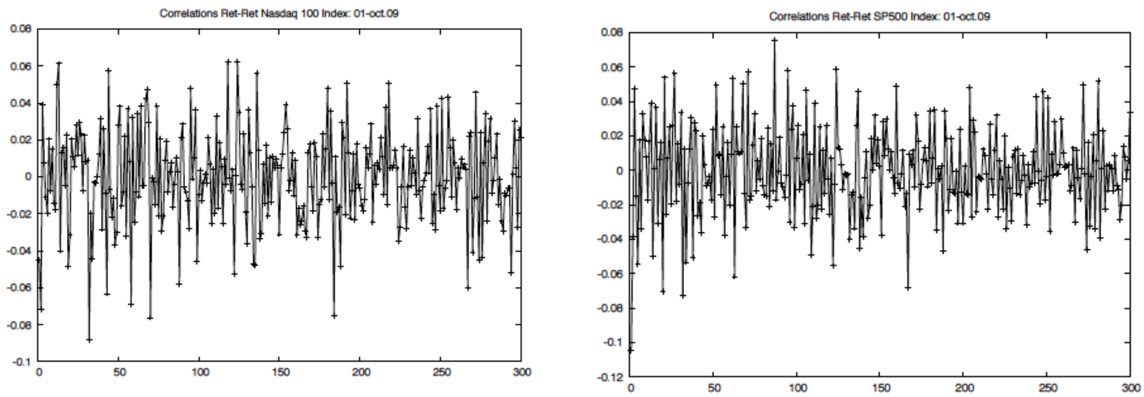
However, the recent market crashes confirmed that this model is far from being suitable. In particular, extreme (or rare) events occur in reality much more often than predicted by such models: with such models, the probability of a crash is so small that the crashes of 1987 and 2009 are quite impossible. Several works, in particular Mandelbrot's and Fama's in the sixties, already pointed out the fact that stock process evolution is far from being Gaussian.

A statistical study of financial markets shows that there are some "universal properties" shared by stock or indices, called "stylized facts" of markets. The log-price X_t of a good model should satisfy:

1. stationarity of th returns, ie $(X_t)_t$ has stationary increments,
2. decorrelation: $\mathbb{E}[X_s(X_t - X_s)] = 0$ for $s < t$ (see figure 4)
3. clong-range correlations of the volatility $< X >$ (see figure 5):

$$\mathrm{Corr}\big(< X >_{0,1}, < X >_{t,t+1} \big) = \frac{A}{(1+t)^{\mu}}$$

   with $\mu \in [0; 0.5]$. The limiting case $\mu \to 0$ can also be modelized with correlations of log type: $A - B \ln(1+t)$.



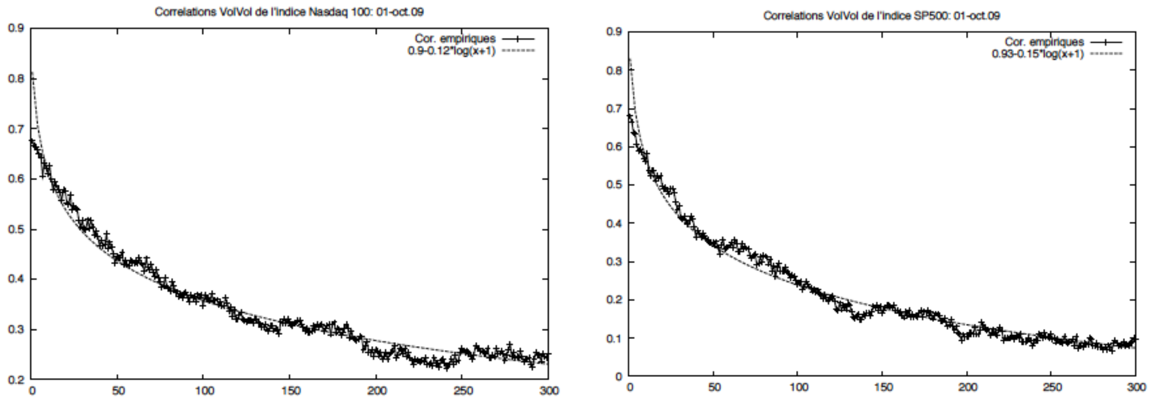
(a) NASDAQ (b) SP500

Figure 4: Daily empirical correlations of indices Nasdaq and SP500 over the period 2001-2009

Mandelbrot's idea is to modelize the evolution of the log price with a MRW (Multifractal Random Walk)

$$X_t = B_{M_t},$$

that is a Brownian motion B seen at the time of an increasing stochastic process M with properties very close to those of the local energy dissipation in turbulence, for instance a Gaussian Multiplicative Chaos. The process M can then be seen as the volatility process and possesses intermittency properties that are close to those observed experimentally (see figure 6).



(a) NASDAQ (b) SP500

Figure 5: Empirical correlations of the volatility of indices Nasdaq and SP500 over the period 2001-2009

2 Convexity inequalities for Gaussian random variables

2.1 Background

We equip $\mathbb{R}^d$ with the inner product

$$\langle x, y\rangle = \sum_{i=1}^{d} x_i y_i.$$

If $X = (X_1, \dots, X_d)$ is a square integrable random vector, the covariance matrix is defined by:

$$K_X = \mathbb{E}\big[(X - \mathbb{E}[X]) \cdot (X - \mathbb{E}[X])^T\big].$$

It is symmetric and positive. If X is Gaussian, its characteristic function is given by:

$$\forall u \in \mathbb{R}^d, \quad \phi_X(u) = \exp\Big(i\langle u, \mathbb{E}[X]\rangle - \frac{1}{2}\langle u, K_X u\rangle\Big).$$

Definition 1. *A stochastic process $(X_t)_{t\in T}$ (where T is an arbitrary index set) is said Gaussian if and only if all its marginals are Gaussian, ie $(X_{t_1}, \dots, X_{t_n})$ is Gaussian for all $t_1, \dots, t_n \in T$.*

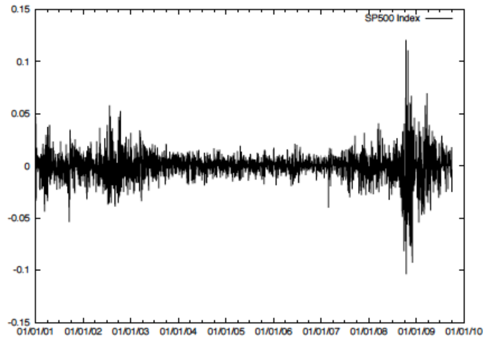
If $(X_t)_{t\in T}$ is a square integrable random process, we define its covariance kernel by:

$$\forall u, v \in T, \quad f_X(u, v) = \operatorname{cov}(X_u, X_v).$$

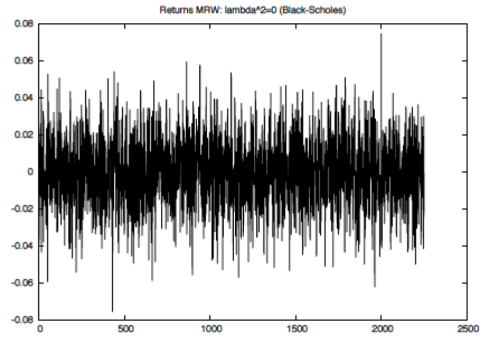
Definition 2. *A stochastic process $(X_t)_{t\in\mathbb{R}^d}$ is called stationary if and only if, for every $z \in \mathbb{R}^d$, both processes $(X_t)_{t\in\mathbb{R}^d}$ and $(X_{z+t})_{t\in\mathbb{R}^d}$ have the same law.*

When X is a stationary Gaussian process, its covariance kernel $f_X(u, v)$ only depends on the difference $u - v$. We can thus write

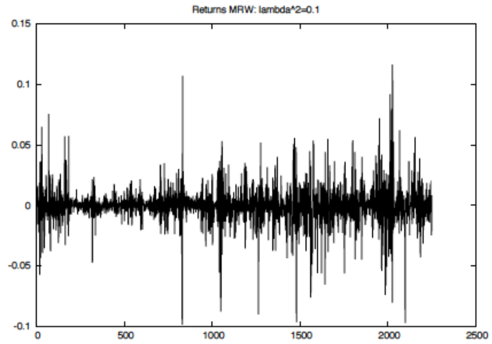
$$f_X(u, v) = g(u - v)$$



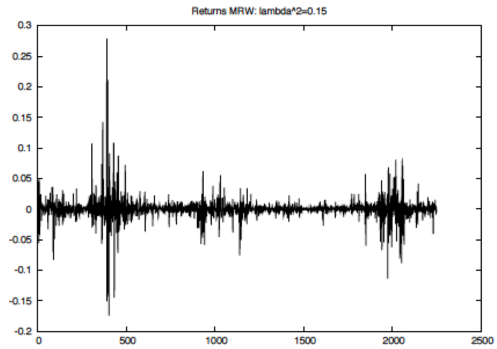
(a) SP500 returns over 2001-2009



(b) Returns simulated with Black-Scholes



(c) Returns simulated with MRW



(d) Returns simulated with MRW

Figure 6: Intermittency in financial markets

for some function g that is necessarily even and of positive type, namely that, for $n \geq 1$, all $t_1, \dots, t_n \in \mathbb{R}^d$ and all $x_1, \dots, x_n \in \mathbb{C}$, we have:

$$\sum_{i,j=1}^{n} x_i \overline{x_j} g(t_i - t_j) \geq 0.$$

The function g is then called covariance kernel of the stationary Gaussian process X.

Theorem 3. *Let f be a real valued of positive type over $\mathbb{R}^d$ and continuous at the origin 0. Then there is a unique measure μ sur $\mathbb{R}^d$, which turns out to be finite and positive, such that*

$$f(t) = \int_{\mathbb{R}^d} e^{i\langle t,x\rangle}\,\mu(dx).$$

When f is integarble then the measure μ possesses a density g with respect to the Lebesgue measure, which is given by

$$g(t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle t,x\rangle} f(t)\,dt.$$

So, when X is a stationary Gaussian process, its covariance kernel is the Fourier transform of a finite positive measure, called the *spectral measure* of the process X.

About existence of stationary Gaussian processes, we claim

Theorem 4. *Let f be a function of positive type over $\mathbb{R}^d$. Then there is a stationary centered Gaussian process X admitting f as spectral measure.*

2.2 Convexity inequalities

Lemma 5. *Let* $(X_i)_{1\leq i\leq n}$ *and* $(Y_i)_{1\leq i\leq n}$ *be two centered independent Gaussian. Let* $(p_i)_{1\leq i\leq n}$ *be a finite family of positive real scalars. If* $\phi : \mathbb{R}_+ \to \mathbb{R}$ *is a smooth function with at most polynomial growth at infinity, we define:*

$$\varphi(t) = \mathbb{E}\big[\phi\big(\sum_{i=1}^{n} p_i e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}\big)\big]$$

with

$$Z_i(t) = \sqrt{t}X_i + \sqrt{1-t}Y_i.$$

Then we have

$$\forall t \in]0,1[, \quad \varphi'(t) = \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \big(\mathbb{E}[X_iX_j] - \mathbb{E}[Y_iY_j]\big)\mathbb{E}\big[e^{Z_i(t)+Z_j(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]-\frac{1}{2}\mathbb{E}[Z_j^2(t)]}\phi''(W_{n,t})\big]$$

where

$$W_{n,t} = \sum_{i=1}^{n} p_i e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}.$$

Preuve. We first differentiate the expression of φ to obtain

$$\varphi'(t) = \mathbb{E}\Big[\frac{1}{2}\sum_{i=1}^{n} p_i\big(\frac{1}{\sqrt{t}}X_i - \frac{1}{\sqrt{1-t}}Y_i - \mathbb{E}[X_i^2] + \mathbb{E}[Y_i^2]\big)e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}\phi'\big(\sum_{i=1}^{n} p_i e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}\big)\Big]$$

Then we use the integration by parts formula for Gaussian random variables:

Lemma 6. *Let* $(X, Y) \in \mathbb{R} \times \mathbb{R}^d$ *be a centered Gaussian random vector and* $f : \mathbb{R}^d \to \mathbb{R}$ *be a smooth function such that* f*, as well as its gradient, has at most polynomial growth at infinity. Then*

$$\mathbb{E}[Xf(Y)] = \mathbb{E}[XY] \cdot \mathbb{E}[\nabla f(Y)].$$

With the help of the lemma, we deduce the following computations for φ':

$$\begin{aligned}
\varphi'(t) =&\mathbb{E}\Big[\frac{1}{2}\sum_{i=1}^n p_i(\frac{1}{\sqrt{t}}X_i - \frac{1}{\sqrt{1-t}}Y_i)e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}\phi'\big(\sum_{i=1}^n p_i e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}\big)\Big]\\
&-\mathbb{E}\Big[\frac{1}{2}\sum_{i=1}^n p_i(\mathbb{E}[X_i^2]-\mathbb{E}[Y_i^2])e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}\phi'\big(\sum_{i=1}^n p_i e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}\big)\Big]\\
=&\frac{1}{2}\sum_{i,j=1}^n p_i p_j \mathbb{E}\Big[(\frac{1}{\sqrt{t}}X_i - \frac{1}{\sqrt{1-t}}Y_i)(\sqrt{t}X_j+\sqrt{1-t}Y_j)\Big]\\
&\times\mathbb{E}\Big[e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}e^{Z_j(t)-\frac{1}{2}\mathbb{E}[Z_j^2(t)]}\phi''\big(\sum_{i=1}^n p_i e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}\big)\Big]\\
&+\frac{1}{2}\sum_{i}^n p_i \mathbb{E}\Big[(\frac{1}{\sqrt{t}}X_i - \frac{1}{\sqrt{1-t}}Y_i)(\sqrt{t}X_i+\sqrt{1-t}Y_i)\Big]\mathbb{E}\Big[e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}\phi'\big(\sum_{i=1}^n p_i e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}\big)\Big]\\
&-\mathbb{E}\Big[\frac{1}{2}\sum_{i=1}^n p_i(\mathbb{E}[X_i^2]-\mathbb{E}[Y_i^2])e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}\phi'\big(\sum_{i=1}^n p_i e^{Z_i(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]}\big)\Big]\\
=&\frac{1}{2}\sum_{i,j=1}^n p_i p_j\big(\mathbb{E}[X_iX_j]-\mathbb{E}[Y_iY_j]\big)\mathbb{E}\big[e^{Z_i(t)+Z_j(t)-\frac{1}{2}\mathbb{E}[Z_i^2(t)]-\frac{1}{2}\mathbb{E}[Z_j^2(t)]}\phi''(W_{n,t})\big],
\end{aligned}$$

which completes the proof. □

Preuve du lemme 6. We first consider the case of independent centered and normalized Gaussian random variables $X \in \mathbb{R}, Y \in \mathbb{R}^d$. Let $A \in M_d(\mathbb{R})$ and $B \in \mathbb{R}^d$. We have:

$$\begin{aligned}
\mathbb{E}[Xf(B\cdot X+AY)] &= \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{R}^d} xf(B\cdot x+Ay)e^{-\frac{1}{2}(x^2+y^2)}\,dxdy\\
&= \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^2} f(B\cdot x+Ay)\partial_x\big(-e^{-\frac{1}{2}(x^2+y^2)}\big)\,dxdy\\
&= \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^2} B\cdot\nabla f(B\cdot x+Ay)e^{-\frac{1}{2}(x^2+y^2)}\,dxdy\\
&= \mathbb{E}[B\cdot\nabla f(B\cdot X+AY)].
\end{aligned}$$

Now we consider the case of a couple $(X', Y') \in \mathbb{R}\times\mathbb{R}^d$ that is centered but not independent. Set

$$X = \frac{X'}{a} \quad \text{and} \quad Y = c^{-1}(Y'-bX')$$

with

$$a = \mathrm{Var}(X')^{1/2}, \quad b = \mathbb{E}[X'Y']/\mathrm{Var}(X'), \quad c = \sqrt{\mathrm{Var}(Y')-\mathbb{E}[X'Y']\mathbb{E}[X'Y']^t/\mathrm{Var}(X')}.$$

Then X, Y are independent centered normalized Gaussian random variables. We deduce:

$$\begin{aligned}
\mathbb{E}[X'f(Y')] &= a\mathbb{E}[Xf(cY+baX)]\\
&= a^2\mathbb{E}[b\cdot\nabla f(baX+cY)],
\end{aligned}$$

which reads

$$\mathbb{E}[X'f(Y')] = \mathbb{E}[X'Y']\cdot\mathbb{E}[\nabla f(Y')].$$

□

Theorem 7. (Comparison principle) *Let $(X_i)_{1\leq i\leq n}$ and $(Y_i)_{1\leq i\leq n}$ be two centered gaussian vectors such that:*

$$\forall i,j,\quad \mathbb{E}[X_iX_j]\leq \mathbb{E}[Y_iY_j].$$

Then:

1. *for each convex function $F:\mathbb{R}_+\to\mathbb{R}$:*

$$\mathbb{E}\big[F\big(\sum_{i=1}^n p_i e^{X_i-\frac{1}{2}\mathbb{E}[X_i^2]}\big)\big]\leq \mathbb{E}\big[F\big(\sum_{i=1}^n p_i e^{Y_i-\frac{1}{2}\mathbb{E}[Y_i^2]}\big)\big],$$

2. *if we further assume*

$$\forall i,\quad \mathbb{E}[X_i^2]=\mathbb{E}[Y_i^2]$$

   then for each increasing function $F:\mathbb{R}_+\to\mathbb{R}$:

$$\mathbb{E}\big[F\big(\sup_{i=1,\dots,n} Y_i\big)\big]\leq \mathbb{E}\big[F\big(\sup_{i=1,\dots,n} X_i\big)\big].$$

Preuve.

1. without loss of generality, we may assume that X and Y are independent. We can then apply the previous lemma and use the fact that the derivative is of negative sign.

2. It is sufficient to treat the case when $F=\mathbb{1}_{]x,\infty[}$ for some constant $x\in\mathbb{R}$. Let β be a postive scalar. We apply 1. to the convex function $\phi(u)=e^{-e^{-\beta x}u}$, to the random vectors $(\beta X_i)_{1\leq i\leq n}$ and $(\beta Y_i)_{1\leq i\leq n}$ and to $p_i=e^{\beta^2\mathbb{E}[X_i^2]/2}$ to obtain:

$$\mathbb{E}\big[e^{-\sum_{i=1}^n e^{\beta(X_i-x)}}\big]\leq \mathbb{E}\big[e^{-\sum_{i=1}^n e^{\beta(Y_i-x)}}\big].$$

   Then we let β go to $+\infty$ to get:

$$\mathbb{P}\big(\sup_{i=1,\dots,n} X_i<x\big)\leq \mathbb{P}\big(\sup_{i=1,\dots,n} Y_i<x\big).$$

□

Corollary 8. *Let $(X_x)_{x\in\mathbb{R}^d}$ and $(Y_x)_{x\in\mathbb{R}^d}$ be two centered Gaussian vectors such that their covariance kernels are continuous and satisfy:*

$$\forall x,x'\in\mathbb{R}^d,\quad f_X(x,x')\leq f_Y(x,x').$$

Let μ be a positive Radon measure over $\mathbb{R}^d$. Then for each convex function $F:\mathbb{R}_+\to\mathbb{R}$, we have:

$$\mathbb{E}\big[F\big(\int_a^b e^{X_t-\frac{1}{2}\mathbb{E}[X_t^2]}\mu(dt)\big)\big]\leq \mathbb{E}\big[F\big(\int_a^b e^{Y_t-\frac{1}{2}\mathbb{E}[Y_t^2]}\mu(dt)\big)\big].$$

Remark. *All the previous results are valid with the same proofs when we do not assume that the considered Gaussian fields are stationary.*

3 Gaussian multiplicative chaos

3.1 Introduction and setup

We define M_+ as the space of positive Radon measures on $\mathbb{R}^d$ for $d \geq 1$. Let p be a real valued continuous function of positive type on $\mathbb{R}^d$. The function p is therefore the covariance kernel of some centered stationary Gaussian process X:

$$\forall x, y \in \mathbb{R}^d, \quad p(x-y) = \mathbb{E}[X_x X_y]. \tag{3}$$

To the process X, we associate the random weight:

$$P(x) = \exp\big(X_x - \frac{1}{2}\mathbb{E}[(X_x)^2]\big) = \exp\big(X_x - \frac{1}{2}p(0)\big). \tag{4}$$

We stress that, in that way, the random weight is normalized:

$$\forall x \in \mathbb{R}^d, \quad \mathbb{E}[P(x)] = 1. \tag{5}$$

Furthermore, for any $\alpha > 0$, we have

$$\mathbb{E}[(P(x))^\alpha] = \exp\big(\frac{1}{2}(\alpha^2 - \alpha)p(0)\big). \tag{6}$$

The random weight P acts on M_+ as follows. To each measure $\sigma \in M_+$, we associate the random Radon measure

$$P\sigma(dx) = P(x)\sigma(dx).$$

Note that for every compact set K of $\mathbb{R}^d$, we have:

$$\mathbb{E}[P\sigma(K)] = \sigma(K).$$

Let us denote by λ the Lebesgue measure on $\mathbb{R}^d$. In order to obtain a stationary random measure, we have to choose $\sigma = \lambda$ (up to a multiplicative constant). Our purpose is to choose the covariance kernel p so as to obtain a nonlinear power law spectrum for the random measure $P\lambda$. For instance, for $d = 1$ and because of the continuity of p at $x = 0$, a straightforward computation shows that:

$$\mathbb{E}[(P\lambda[0,t])^\alpha] \simeq t^\alpha \mathbb{E}[(P(0))^\alpha] = t^\alpha \exp\big(\frac{1}{2}(\alpha^2 - \alpha)p(0)\big) \quad \text{as } t \to 0.$$

The power law spectrum is linear: the limit as t goes to 0 of the quantity

$$\frac{\ln \mathbb{E}[(P\lambda[0,t])^\alpha]}{\ln t}$$

is linear w.r.t. α. This argument shows that the above construction is not general enough. We must be able to deal with covariance kernels p that may diverge around the origin. However, such kernels are associated to Gaussian processes with infinite variance! That can be done by considering X not as a pointwise defined process but as a random distribution: X is a Gaussian distribution such that for each smooth functions φ, ψ with compact support

$$\mathbb{E}[X(\varphi)X(\psi)] = \int \varphi(x)\psi(y)p(x-y)\,dxdy.$$

The main difficulty is now to define random measures that can be formally understood as

$$K \mapsto \int_K \exp\big(X_x - \frac{1}{2}\mathbb{E}[(X_x)^2]\big)\,dx, \tag{7}$$

because we must give sense to the exponential of a distribution, which is not straightforward. This is the main purpose of the theory of Gaussian multiplicative chaos.

Remark. *A particular type of Gaussian distribution has recently received much attention: the Gaussian Free Field. When X is a GFF, the formal measure defined by 7 is called the Quantum measure (see [7, 17]) in the context of the KPZ formula [12].*

The main idea of the theory is to assume that the "possibly divergent" kernel p can be written as a sum of positive continuous covariance kernels p_n on $\mathbb{R}^d$:

$$\forall x \in \mathbb{R}^d, \quad p(x) = \sum_{n=1}^{+\infty} p_n(x). \tag{8}$$

Note that the sum perfectly makes sense as the functions p_n are positive. Such kernels p are said to be of σ-positive type. Each p_n thus can be seen as the covariance kernel of a stationary centered Gaussian process X^n and, without loss of generality, we may assume that the processes $(X^n)_n$ are independent. For $n \in \mathbb{N}$, we define the stationary centered Gaussian process

$$\forall x \in \mathbb{R}^d, \quad Y_x^n = X_x^1 + X_x^2 + \cdots + X_x^n$$

with associated covariance kernel

$$q_n(x) = p_1(x) + p_2(x) + \cdots + p_n(x). \tag{9}$$

To the process Y^n, we associate the random weight

$$Q^n(x) = \exp\big(Y_x^n - \frac{1}{2}\mathbb{E}[(Y_x^n)^2]\big) = \exp\big(Y_x^n - \frac{1}{2}q_n(0)\big).$$

For each Radon measure $\sigma \in M_+$ and $n \geq 1$, we define the Radon random measure

$$(Q^n\sigma)(dx) = Q^n(x)\sigma(dx).$$

For each compact set K of $\mathbb{R}^d$, we have:

$$\mathbb{E}[Q^n\sigma(K)] = \sigma(K).$$

Actually, it turns out that the sequence $(Q^n\sigma(K))_n$ is a martingale. Let $\mathcal{F}_n$ be the sigma field generated by the processes $\{X_u^k; k \leq n, u \in \mathbb{R}^d\}$. For each bouded Borelian set A of $\mathbb{R}^d$, we have

$$\mathbb{E}[Q^n\sigma(A)|\mathcal{F}_{n-1}] = \int_A Q^{n-1}(x)\mathbb{E}[\exp\big(X_x^n - \frac{1}{2}\mathbb{E}[(X_x^n)^2]\big)]\sigma(dx) = Q^{n-1}\sigma(A).$$

The sequence $(Q^n\sigma(A))_n$ is a positive martingaleand therefore converges almost surely towards a positive random variable denoted by $Q\sigma(A)$. It can be checked (use the Caratheodory's extension theorem) that, almost surely, the sequence of measure $(Q^n\sigma(A))_n$ weakly converges towards a positive Radon measure on $\mathbb{R}^d$, denoted by $Q\sigma$. The linear operator $Q : \sigma \in M_+ \mapsto Q\sigma$ is called operator of multiplicative chaos associated to the kernel p. We will see that this operator does not depend on the decomposition (8) of p.

In what follows, we are especially interested in the case when

$$p(x) = \gamma^2 \ln_+ \frac{T}{|x|} + g(x)$$

where g is a continuous bounded function on $\mathbb{R}^d$. The parameter γ^2 is often called *intermittency parameter*.

3.2 Support et degeneracy

Let B be a non empty ball of $\mathbb{R}^d$ and $\sigma \in M_+$. It is plain to see that the event $\{Q\sigma(B) > 0\}$ is an event of the asymptotic sigma field generated by the processes $(X^n)_n$. It thus has probability 0 or 1. By considering an exhaustive sequence of balls of $\mathbb{R}^d$, we deduce

Proposition 9. *The event $\{Q\sigma \equiv 0\}$ has probability 0 or 1.*

In the same spirit, we have for the Lebesgue measure λ:

Proposition 10. *Either of the following situations occurs with probability 1:*

1. *almost surely, $Q\lambda \equiv 0$.*
2. *for every ball B of $\mathbb{R}^d$,*

$$Q\sigma(B) > 0.$$

In particular, when it is non trivial, the random measure $Q\lambda$ admits $\mathbb{R}^d$ as support.

Definition 11.
- *If $I\!P(Q\sigma \equiv 0) = 1$, we will say that Q is degenerated at σ, otherwise we will say that Q is non degenerated at σ.*
- *Consider a bounded Borelian set A of $\mathbb{R}^d$. Since $(Q^n\sigma(A))_n$ is a positive martingale, it is regular (ie converges in L^1) if and only if $\mathbb{E}[Q\sigma(A)] = \sigma(A)$. In the case when $(Q^n\sigma(A))_n$ is regular for each bounded Borelian set A of $\mathbb{R}^d$, we will say that Q is strongly non degenerated at σ.*

For $\sigma \in M_+$, we define the postive Radon random measures:

$$\sigma_0(A) = \mathbb{E}[Q\sigma(A)], \quad \sigma_1(A) = \sigma(A) - \sigma_0(A).$$

Note that we have:

$$\mathbb{E}[Q\sigma(A)|\mathcal{F}_n] = Q^n\sigma_0(A),$$

so that $(Q^n\sigma_0(A))_n$ is a regular martingale. We deduce that σ can be decomposed in two measures $\sigma = \sigma_0 + \sigma_1$ such that

$$\mathbb{E}[Q\sigma_0(A)] = \sigma_0(A), \quad \mathbb{E}[Q\sigma_1(A)] = 0.$$

Put in other words, we have proved

Theorem 12. *Each measure $\sigma \in M_+(\mathbb{R}^d)$ can be decomposed into the sum of two measures $\sigma_0, \sigma_1 \in M_+(\mathbb{R}^d)$ such that:*

- *Q is degenerated at σ_1,*
- *Q is strongly non-degenerated at σ_0.*

3.3 Uniqueness

We have just seen how to define a Gaussian multiplicative chaos

$$K \mapsto \int_K \exp\left(X_x - \frac{1}{2}\mathbb{E}[(X_x)^2]\right) dx,$$

where X is a Gaussian distribution with covariance kernel

$$p(x) = \sum_{n=1}^{+\infty} p_n(x).$$

The first important question that we have to answer is the following: what happens if we have two decompositions

$$p(x) = \sum_{n=1}^{+\infty} p_n(x) = \sum_{n=1}^{+\infty} p'_n(x)$$

of the kernel p? Are the laws of the resulting random measures identical? Of course, we will see that the answer is positive and this is the cornerstone of the theory.

Theorem 13. *Choose $\sigma \in M_+$. Let $(p_n)_n$ and $(p'_n)_n$ be two families of continuous positive covariance kernels such that*

$$\sum_n p_n(x) = \sum_n p'_n(x).$$

The associated multiplicative chaos $Q\sigma$ and $Q'\sigma$ have the same law.

Sketch of proof.
Step 1: prove that the chaos operators Q and Q' are degenerated/strongly non degenerated at the same measures (we do not give the proof but it relies on Corollary 8 in a way similar to that explained in step 2).
Step 2: Let K be a compact subset of $\mathbb{R}^d$. Because of the step 1, we may assume that both martingales $(Q_n\sigma(K))_n$ and $(Q'_n\sigma(K))_n$ are regular.

Fix $N \in \mathbb{N}^*$. Denote by $(q_n)_n$ and $(q'_n)_n$ the sequences of partial sums. The sequence $(q_n - q'_N)_n$ converges increasingly towards $q - q'_N$, which is a positive function. Therefore the negative part $((q_n - q'_N)_-)_n$ converges decreasingly towards 0. Dini's theorem ensures that this latter convergence is uniform. So, for a given $\epsilon > 0$ and n large enough, we have

$$q'_N(x) \leq q_n(x) + \epsilon$$

over K. Thus we can apply Corollary 8 to obtain

$$\mathbb{E}\Big[F\Big(Q'_N\sigma(A)\Big)\Big] \leq \mathbb{E}\Big[F\Big(Q_n\sigma(A)e^{\sqrt{\epsilon}Y - \frac{\epsilon}{2}}\Big)\Big]$$

for every Borelian subset $A \subset K$, F convex function and Y centered normalized Gaussian variable independent of Q_n. If we further assume that F is Lipschitzian (with Lipschitz constant L) we have

$$\begin{aligned}\mathbb{E}\Big[F\Big(Q'_N\sigma(A)\Big)\Big] \leq& \mathbb{E}\Big[F\Big(Q_n\sigma(A)e^{\sqrt{\epsilon}Y-\frac{\epsilon}{2}}\Big)\Big]\\ \leq& \mathbb{E}\Big[F\Big(Q_n\sigma(A)\Big)\Big] + L\mathbb{E}\Big[Q_n\sigma(A)|e^{\sqrt{\epsilon}Y-\frac{\epsilon}{2}}-1|\Big]\\ =& \mathbb{E}\Big[F\Big(Q\sigma(A)\Big)\Big] + L\lambda(A)\mathbb{E}\Big[|e^{\sqrt{\epsilon}Y-\frac{\epsilon}{2}}-1|\Big].\end{aligned}$$

In the last equality, we have used the fact that (see Corollary 8)

$$\mathbb{E}\Big[F\Big(Q\sigma(A)\Big)\Big] = \lim_{n\to\infty}\uparrow \mathbb{E}\Big[F\Big(Q_n\sigma(A)\Big)\Big].$$

For the same reason, we can take the limit as N goes to infinity. Since ϵ is arbitrary, we deduce

$$\mathbb{E}\Big[F\Big(Q'\sigma(A)\Big)\Big] \leq \mathbb{E}\Big[F\Big(Q\sigma(A)\Big)\Big].$$

Of course, the symmetric argument gives us the reversed inequality so that we can place $=$ in the above relation. We deduce that $Q'(A)$ and $Q(A)$ have the same law. We can carry out the same job for the sums

$$\sum_{i=1}^n \lambda_i Q(A_i) \quad \text{and} \quad \sum_{i=1}^n \lambda_i Q'(A_i)$$

with $\lambda_i \geq 0$. This shows that the random measures $Q\sigma$ and $Q'\sigma$ have the same law. □

3.4 Multifractality, stochastic scale invariance

Now we investigate the action of the chaos operator Q on the Lebesgue measure λ. We stress that the random measure $Q\lambda$ is stationary, namely that its law is invariant under translations. The motivation to study the Lebesgue measure, beyond the stationarity, is the fact that the Lebesgue measure possesses scaling properties. Well chosen chaos then inherit some nice stochastic scaling properties that we are going to explain below.

We assume that the covariance kernel p of the chaos operator is given by

$$p(x) = \gamma^2 \ln_+ \frac{1}{|x|} + g(x) \tag{10}$$

where g is a continuous bounded function on $\mathbb{R}^d$. Kahane [11] proved that:

Theorem 14. *(Kahane 1985)*
- if $\gamma^2 \geq 2d$, the associated chaos operator (10) is degenerated at the Lebesgue measure.
- if $\lambda^2 < 2d$, the associated chaos operator (10) is strongly non degenerated at the Lebesgue measure.

Remark. *-For instance, it is plain to see that all kernels p of the form*

$$p(x) = \int_1^{+\infty} \frac{k(u|x|)}{u}\,du,$$

where k is a continuous positive covariance kernel, can be rewritten as in (10). The intermittency parameter γ^2 is then given by $k(0)$.
-When X is a Gaussian Free Field on a smooth domain $D \subset \mathbb{R}^2$ grounded at 0 on the boundary of D, its covariance kernel is given by

$$p(x,y) = G(x,y)$$

where G is the Green function of the Laplacian on D, ie a distributional solution of the equation

$$\triangle G(x,\cdot) = -2\pi\delta_x.$$

Then G can be rewritten as

$$G(x,y) = 2\pi\int_0^{+\infty} p_D(t,x,y)\,dt$$

where p_D are the transition densities of the symmetric semi-group $(P_t)_t$ defined by

$$\partial_t P_t f = \triangle P_t f \text{ on } \boldsymbol{D}, \quad P_t f \in H_0^1(D).$$

It is plain to see that p_D is given by

$$p_D(t,x,y) = I\!\!P_x(B_t \in dy, \tau_D > t),$$

where B is a standard 2-dimensional Brownian motion and τ_D the first exit time of B from the domain D. G is of σ positive type since it can be rewritten as

$$G(x,y) = \sum_{n\in\mathbb{N}} p_n(x,y)$$

with

$$p_n(x,y) = 2\gamma^2\pi\int_{2^{-n-1}}^{2^{-n}} p_D(t,x,y)\,dt + 2\gamma^2\pi\int_{2^n}^{2^{n+1}} p_D(t,x,y)\,dt.$$

We choose such a decomposition to get rid of the possible singularities at $t = 0$ ot $t = +\infty$. The kernels p_n are continuous, positive (both because p_D is) and of positive type since

$$\begin{aligned}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)f(y)\int_a^b p_D(t,x,y)\,dt\,dxdy &= \int_a^b\int_{\mathbb{R}^2} f(x)P_t f(x)\,dxdt \\ &= \int_a^b\int_{\mathbb{R}^2} P_{t/2}f(x)P_{t/2}f(x)\,dxdt \\ &\geq 0.\end{aligned}$$

Theorem 15. *Consider a covariance kernel p given by (10) for some $\lambda^2 < 2d$. The chaos (w.r.t. the Lebesgue measure) $Q\lambda$ possesses a nonlinear power law spectrum: for any $p \geq 0$ such that $Q\lambda$ admits a moment of order p (i.e. for $p \in [0, \frac{2d}{\gamma^2}[$ see theorem 18), there is a constant $D = D(p,A) > 0$ such that*

$$\mathbb{E}\big[Q\lambda(tA)^p\big] \simeq Dt^{\xi(p)} \quad \text{as } t \to 0$$

where A is a bounded open set, and

$$\forall p \in \mathbb{R}, \quad \xi(p) = \big(d + \frac{\lambda^2}{2}\big)p - \frac{\lambda^2}{2}p^2. \tag{11}$$

Proof. The proof is left as an exercise. We just give an hint. First note that the theorem is easy to prove when $g \equiv 0$ and then use Corollary 8 to deduce the general case. □

Remark. *Note that the power law spectrum in* (11) *is quadratic. This is because the involved processes are Gaussian. It is possible to generalize the theory of multiplicative chaos to infinitely divisible processes so as to obtain nonlinear power law spectrum of the type*

$$\forall p \in \mathbb{R}, \quad \xi(p) = dp - \psi(p)$$

where ψ is the characteristic function of any infinitely divisible law such that $\psi(1) = 0$.

We now focus on the situation when the function g is constant on a neighborhood of 0, or even on the whole space $\mathbb{R}^d$. As we will see, the chaos measure then possesses a nice stochastic self-similarity property called stochastic scale invariance. We first justify the fact that we can choose g constant on a neighborhood of 0.

Theorem 16. *For $d \leq 2$, the function*

$$x \mapsto \lambda^2 \ln_+ \frac{T}{|x|} \tag{12}$$

is of σ-positive type. For $d = 3$, it is an open question to know if the kernel (12) *is of σ-positive type and for $d \geq 4$, it is not even of positive type. Nevertheless, for $d \geq 3$, we can find an isotropic function g that is constant on a neighborhood of 0 and such that the kernel*

$$p(x) = \gamma^2 \ln_+ \frac{T}{|x|} + g(x) \tag{13}$$

is of σ-positive type.

Proof. A straightforward computation yields:

$$\ln_+ \frac{T}{|x|} = \int_0^{+\infty} (t - |x|)_+ \nu_T(dt)$$

where ν_T is the measure (δ_T is the Dirac mass at T):

$$\nu_T(dt) = \mathbb{1}_{[0,T]}(t) \frac{dt}{t^2} + \frac{1}{T}\delta_T(dt).$$

Hence for any $\mu > 0$, we have:

$$\ln_+ \frac{T}{|x|} = \frac{1}{\mu} \ln_+ \frac{T^\mu}{|x|^\mu} = \int_0^{+\infty} (t - |x|^\mu)_+ \nu_{T^\mu}(dt).$$

We are thus led to consider the possible values of $\mu > 0$ such that the function $(1 - |x|^\mu)_+$ is of positive type: this is the Kuttner-Golubov problem (see [10]).

For $d = 1$, it is straightforward to see that $(1 - |x|)_+$ is of positive type (compute the inverse Fourier transform). In dimension 2, Pasenchenko [14] proved that the function $(1 - |x|^{1/2})_+$ is of positive type on $\mathbb{R}^2$. We can thus write

$$\gamma^2 \ln_+ \frac{T}{|x|} = \sum_{n \geq 1} p_n(x)$$

with

$$p_n(x)=\int_{\frac{1}{n}}^{\frac{1}{n-1}}(t-|x|^{\mu})_+\nu_{T^\mu}(dt)$$

with $\mu=1$ in dimension 1 and $\mu=1/2$ in dimension 2.

We now focus on the second part ot the theorem and consider $d\geq 3$. Let us denote by S the sphere of $\mathbb{R}^d$ and by σ the unique uniform measure on the sphere such that $\sigma(S)=1$. In particular, the reader is reminded that this measure is invariant under rotations. We define the function

$$F(x)=\gamma^2\int_S \ln_+\frac{T}{|\langle x,s\rangle|}\sigma(ds). \tag{14}$$

As σ is unvariant under rotations, the function F is isotrop. Let us compute F on the neighborhood of 0. Fix $x\in\mathbb{R}^d$ such that $|x|\leq T$. Write $x=|x|e$ where $e\in S$. Then we have

$$F(x)=\gamma^2\int_S \ln\frac{T}{|x||\langle e,s\rangle|}\sigma(ds)=\gamma^2\ln\frac{T}{|x|}+\gamma^2\int_S\ln\frac{1}{|\langle e,s\rangle|}\sigma(ds).$$

Note that, by invariance under rotations, the second term matches $\gamma^2\int_S\ln\frac{1}{|\langle e_1,s\rangle|}\sigma(ds)$ (where e_1 is any fixed vector of the sphere) and is therefore constant (it does not depend on x). Finally, we just stress that the integral $\gamma^2\int_S\ln\frac{1}{|\langle e_1,s\rangle|}\sigma(ds)$ is finite: indeed, under the probability measure σ, the random variable $|\langle e_1,s\rangle|$ admits a density w.r.t. the Lebesgue measure given by

$$\frac{2\Gamma(\frac{d}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{d-1}{2})}(1-y^2)^{\frac{d-3}{2}}\mathbb{1}_{[0,1]}(y)\,dy.$$

□

Theorem 17. *Let p be a covariance kernel of σ-positive type given by*

$$p(x)=\gamma^2\ln_+\frac{T}{|x|}+g(x) \tag{15}$$

where g is a continuous bounded function over $\mathbb{R}^d$, constant over a ball $B(0,T)$. Then the associated chaos $Q\lambda$ is **stochastically scale invariant***:*

$$\forall\, 0<\alpha<1,\quad (Q\lambda(\alpha A))_{A\subset B(0,T)}\stackrel{law}{=}\alpha^d e^{Y_\alpha-\frac{1}{2}\mathbb{E}[Y_\alpha^2]}(Q\lambda(A))_{A\subset B(0,T)},$$

where Y_α is a Gaussian random variable, independent of the measure $(Q\lambda(A))_{A\subset B(0,T)}$, with mean 0 and variance $\gamma^2\ln\frac{1}{\alpha}$.

Proof. Let

$$p(x)=\sum_{n=1}^{+\infty}p_n(x)$$

be a decomposition of p into a sum of positive continuous covariance kernels. We stick to the notations introduced in subsection 3.1. For any Borelian set $A\subset B(0,T)$, we have:

$$\begin{aligned}Q^n\lambda(\alpha A)&=\int_{\alpha A}e^{Y_x^n-\frac{1}{2}\mathbb{E}[(Y_x^n)^2]}\,dx\\&=\alpha^d\int_A e^{Y_{\alpha z}^n-\frac{1}{2}\mathbb{E}[(Y_{\alpha z}^n)^2]}\,dz.\end{aligned}$$

By taking the limit as $n \to \infty$, we can see that $Q\lambda(\alpha\cdot)$ is, up to the multiplicative factor α^d, a Gaussian multiplicative chaos with kernel $p(\alpha x)$ on $B(0,T)$. Furthermore, for $x \in B(0,T)$ we have

$$p(\alpha x) = \gamma^2 \ln \frac{1}{\alpha} + \gamma^2 \ln_+ \frac{T}{|x|} + g(\alpha x) = \gamma^2 \ln \frac{1}{\alpha} + p(x).$$

Such a chaos can be obtained as the limit

$$\lim_{n\to\infty} \alpha^d \int_A e^{Y_x^n - \frac{1}{2}\mathbb{E}[(Y_x^n)^2] + Y_\alpha - \frac{1}{2}\mathbb{E}[(Y_\alpha)^2]} \, dx = \alpha^d e^{Y_\alpha - \frac{1}{2}\mathbb{E}[(Y_\alpha)^2]} Q\lambda(A)$$

where Y_α is a Gaussian random variable with mean 0 and variance $\lambda^2 \ln \frac{1}{\alpha}$, independent of $Q\lambda$. Due to the uniqueness property of the law of the chaos, the result follows. □

3.5 Moments

Theorem 18. *Let $Q\lambda$ be the multiplicative chaos asociated to the Lebesgue measure and to the kernel p of the form (10). If $\gamma^2 < 2d$ then we have for $q > 1$:*

$$\forall K \text{ compact set}, \quad \mathbb{E}[Q\lambda(K)^q] < +\infty \iff \gamma^2 < \frac{2d}{q}. \tag{16}$$

Proof. The complete proof of the theorem can be found in [11]. To have a rough idea of the reasons that make this theorem is valid, we will just prove in dimension $d = 1$

$$\forall K \text{ compact set}, \quad \mathbb{E}[Q\lambda(K)^q] < +\infty \Longrightarrow \gamma^2 < \frac{2d}{q}.$$

Fix $n \in \mathbb{N}$. From the super-additivity of the mapping $x \in \mathbb{R}_+ \mapsto x^q$ and the stationarity of the measure $Q\lambda$, we have

$$\mathbb{E}\left[Q\lambda[0;1]^q\right] = \mathbb{E}\left[\left(Q\lambda[0;\frac{1}{n}] + Q\lambda[\frac{1}{n};\frac{2}{n}] + \cdots + Q\lambda[\frac{(n-1)}{n};1]\right)^q\right] \tag{17}$$

$$\geq \mathbb{E}\left[\left(Q\lambda[0;\frac{1}{n}]\right)^q + \left(Q\lambda[\frac{1}{n};\frac{2}{n}]\right)^q + \cdots + \left(Q\lambda[\frac{(n-1)}{n};1]\right)^q\right] \tag{18}$$

$$= n\mathbb{E}\left[\left(Q\lambda[0;\frac{1}{n}]\right)^q\right] \tag{19}$$

We can use Theorem 15 to obtain

$$\mathbb{E}\left[\left(Q\lambda[0;\frac{1}{n}]\right)^q\right] \geq D n^{-\xi(q)} \tag{20}$$

Necessarily, we deduce

$$1 - \xi(q) \leq 0,$$

which reads $\gamma^2 \leq \frac{2}{q}$. □

3.6 A few words about financial applications

In dimension 1, we can choose the kernel

$$p(x) = \gamma^2 \ln_+ \frac{T}{|x|}$$

for some $\gamma^2 < 2$. It is common in finance to denote by M the increasing process associated to the multiplicative chaos applied to the Lebesgue measure:

$$\forall t \geq 0, \quad M_t \overset{notation}{=} Q\lambda([0,t]).$$

Mandelbrot suggested to change the time of a Brownian motion B with M to obtain a mathematical model for the (log) price of the stocks/indices ($\sigma > 0$ is a parameter):

$$X_t = \sqrt{\sigma} B_{M_t}.$$

On the financial markets, the observed values of the parameters for indices are closed to $\sigma \simeq 10^{-2}$, $\lambda^2 \simeq 0,03$ and $T \simeq 5 - 10$ years (there is actually a challenging theoretical problem in estimating T).

The process X possesses many properties observed on the market:

- X is a square integrable continuous martingale. The volatility is then defined as the quadratic variations of the martingale, that it $< X >_t = \sigma M_t$.
- Long-range correlations of the volatility:

$$\mathrm{Cov}(M_1, M_{t+1} - M_t) = \int_0^1 \int_t^{t+1} (e^{\lambda^2 \ln_+ \frac{T}{|r-u|}} - 1)\, dr\, du \simeq \frac{T^{\lambda^2}}{t^{\lambda^2}}$$

  for $1 \ll t \ll T$.
- fat tails distribution...

In comparison with local volatility models, an important advantage of Mandelbrot's approach is also the simplicity of the model (3 parameters to estimate).

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