Algebraic aspects of quantum integrable spin chains

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Introduction to quantum spin chains

One dimensional (discretized) quantum mechanical models, local Hilbert space is an irreducible module of an associative algebra (e.g. $U(\mathfrak{J})$, where $\mathfrak{J}$ is a Lie algebra). A prototype – the Heisenberg (anti)ferromagnet model of N spins with a nearest neighbour interaction:

$$\text{periodic b.c.:} \quad H = J \sum_{n=1}^{N} \sigma_n^a \sigma_{n+1}^a \,, \qquad \text{open b.c.:} \quad H = J \sum_{n=1}^{N-1} \sigma_n^a \sigma_{n+1}^a \,.$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

H acts on $\mathcal{V} = \left(\mathbb{C}^2\right)^{\otimes N}$.

$$\sigma_n^a \equiv I^{\otimes n-1} \otimes \sigma^a \otimes I^{\otimes N-n}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The underlying symmetry is the algebra $A = U(su_2)$:

$$[E, F] = H, \qquad [H, E] = E, \qquad [H, F] = -F.$$

An irreducible finite dimensional representation S has the highest weight vector $v \in S$: $Ev = 0, \quad Hv = \Lambda v, \quad \Lambda \in \mathbb{N}_0$. We have $S = A.v$.

the trivial representation :	the fundamental representation :
$\Lambda = 0$	$\Lambda = 1, \qquad H \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^3,$
$E, F, H \to 0$	$E \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$

Global action of A on $\mathcal{V}$:

$$\mathcal{E} = \sum_{n=1}^{N} E_n, \quad \mathcal{F} = \sum_{n=1}^{N} F_n, , \quad \mathcal{H} = \sum_{n=1}^{N} H_n,$$

$$a_n \equiv \sum_{n=1}^{N} I^{\otimes n-1} \otimes a \otimes I^{\otimes N-n}$$

Properties of the Heisenberg Hamiltonian:

a) locality – $H = \sum_n H_{n,n+1}$;

b) A – invariance: $[H_{n,n+1}, \mathcal{E}] = [H_{n,n+1}, \mathcal{F}] = [H_{n,n+1}, \mathcal{H}] = 0$;

c) integrability: H belongs to a complete set of mutually commuting Hermitian operators acting on $\mathcal{V}$.

Origin of the integrability – existence of the R–matrix:

$$R(\lambda) = \begin{pmatrix} 1+\lambda & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 1+\lambda \end{pmatrix}$$
$$= (1 + \frac{1}{2}\lambda)\, I \otimes I + \lambda\, \sigma^a \otimes \sigma^a \qquad \in \mathrm{End}\,\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)$$
$$= I \otimes I + \lambda\, \mathbb{P}$$

permutation $\qquad \mathbb{P}\, x \otimes y = y \otimes x, \qquad \forall x, y \in \mathbb{C}^2$

Properties of the R–matrix:

a) Yang–Baxter equation in End $\left(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2\right)$:

$$R_{12}(\lambda-\mu)\, R_{23}(\lambda)\, R_{12}(\mu) = R_{23}(\mu)\, R_{12}(\lambda)\, R_{23}(\lambda-\mu)$$

b) regularity: $R(0) = I \otimes I$.

Yang–Baxter $\Rightarrow$ the trace of the monodromy matrix generates a set of mutually commuting "quantum integrals of motion" $\{Q_k\}$:

$$\hat{R}(\lambda) \equiv \mathbb{P}\, R(\lambda), \qquad T_a(\lambda) = \hat{R}_{a,N}(\lambda) \ldots \hat{R}_{a,1}(\lambda) \quad \text{End}\left(\mathbb{C}^2 \otimes \left(\mathbb{C}^2\right)^N\right)$$

$$\text{YB} \Rightarrow \quad R_{ab}(\lambda-\mu)\, T_a(\lambda) \otimes_{ab} T_b(\mu) = T_b(\mu) \otimes_{ab} T_a(\lambda)\, R_{ab}(\lambda-\mu)$$

$$\Rightarrow \quad \left[\operatorname{tr} T(\lambda), \operatorname{tr} T(\mu)\right] = 0$$

$$\Rightarrow \quad \operatorname{tr} T(\lambda) = \sum_k \lambda^k Q_k, \qquad \left[Q_k, Q_m\right] = 0$$

Regularity, $R(0) = I \otimes I \Rightarrow$ locality of H:

$$H \sim \left(\mathrm{tr}\, T(0)\right)^{-1} \left(\frac{d}{d\lambda} \mathrm{tr}\, T(\lambda)\right)_{\lambda=0} = \sum_{n=1}^{N} \left(\frac{d}{d\lambda} R_{n,n+1}(\lambda)\right)_{\lambda=0}$$

$$= \frac{1}{2} \sum_{n=1}^{N} \left(I \otimes I + \sigma_n^a \sigma_{n+1}^a\right) = \mathbb{P}_{n,n+1}$$

A – invariance:

$$H_{n,n+1} = \mathbb{P}_{n,n+1} \quad \Rightarrow \quad [H_{n,n+1}, a_n + a_{n+1}] = 0$$

$\Rightarrow$ H commutes with $\mathcal{E}$, $\mathcal{F}$, $\mathcal{H}$.

Integrability $\Rightarrow$ the Heisenberg chain is exactly solvable.
The Bethe ansatz: the spectrum of H is given

$$E(\lambda_1, \ldots, \lambda_M) = \sum_{k=1}^{M} \frac{1}{1-\lambda_k^2}, \qquad \left(\frac{\lambda_k + 1}{\lambda_k - 1}\right)^N = \prod_{m \neq k}^{M} \frac{\lambda_m - \lambda_k + 2}{\lambda_m - \lambda_k - 2}$$

Algebraic Bethe ansatz (exploits existence of a local highest weight vector), Baxter Q–operator (more general),…

$A = U(su_2)$, $\Lambda = 2$: integrable A–invariant nearest neighbour Hamiltonians ($\Leftrightarrow$ there exists a regular $R(\lambda)$):

$$H_{n,n+1} = \left(\vec{S}_n \cdot \vec{S}_{n+1}\right)^2 + \alpha \left(\vec{S}_n \cdot \vec{S}_{n+1}\right),$$

for $\alpha = -1, 0, 1$.

In general, $H_{n,n+1}$ is a polynomial in $\left(\vec{S}_n \cdot \vec{S}_{n+1}\right)$ of degree Λ.

Further generalizations:
one can search for a regular $R(\lambda) \in \mathrm{End}\left(S \otimes S\right)$
a) for irreducible representations S of a higher weight Λ;
b) for a deformed symmetry A_q, e.g. the quantum group case;
c) for a higher rank Lie algebra A or its quantum deformation A_q.

Relation to 2D statistical models...

Generalization

A quantum spin chain with the local Hamiltonian density

$$H_{n,n+1} = R'_{n,n+1}(0)$$

is defined by a regular solution of the Yang–Baxter equation, $R(\lambda) \in \mathrm{End}\big(S \otimes S\big)$, where S is an irreducible highest weight module of an associative bialgebra A.

multiplication :	comultiplication :
$m : A \otimes A \to A$	$\Delta : A \to A \otimes A$
associativity :	coassociativity :
$(a\,b)\,c = a\,(b\,c)$	$(\Delta \otimes id)\Delta(a) = (id \otimes \Delta)\Delta(a)$

A is a bialgebra if $\Delta(a\,b) = \Delta(a)\,\Delta(b) \quad \forall a, b.$

Example: $U(su_2)$ is a bialgebra, $\Delta(a) = a \otimes I + I \otimes a$ for $a = E, F, H$. This case is cocommutative, $\Delta'(a) \equiv \mathbb{P}\Delta(a)\mathbb{P} = \Delta(a) \;\; \forall a$.

A noncocommutative deformation of $U(su_2)$ is the "quantum" algebra $U_q(su_2)$:

$$[E,F] = \frac{q^H - q^{-H}}{q - q^{-1}}, \qquad [H,E] = E, \qquad [H,F] = -F.$$

$$\Delta(E) = E \otimes q^{-\frac{1}{2}H} + q^{\frac{1}{2}H} \otimes E,$$
$$\Delta(F) = F \otimes q^{-\frac{1}{2}H} + q^{\frac{1}{2}H} \otimes F,$$
$$\Delta(H) = H \otimes I + I \otimes H.$$

For generic q (e.g. $q > 1$), the comultiplication determines the Clebsch–Gordan decomposition:

$$S(\Lambda) \otimes S(\Lambda) = \bigoplus_{\Lambda'=0}^{\Lambda} S(2\Lambda')$$

Global action of $U_q(su_2)$ on $S(\Lambda)^{\otimes N}$:

$$\mathcal{E} = \Delta^{N-1}(E) = \sum_{n=1}^{N} q^{-\frac{1}{2}H_1} \otimes \ldots \otimes q^{-\frac{1}{2}H_{n-1}} \otimes E \otimes q^{\frac{1}{2}H_{n+1}} \otimes \ldots \otimes q^{\frac{1}{2}H_N}$$

$U_q(su_2)$–invariant $R(\lambda)$

For generic q, the center of $U_q(su_2)$ is generated by the Casimir element

$$C = F\,E + \frac{q\,q^H + q^{-1}q^{-H}}{(q - q^{-1})^2}$$

$U_q(su_2)$–invariant $R(\lambda)$:

$$R(\lambda) = R(\lambda, \Delta(C)) = \sum_{\Lambda'=0}^{\Lambda} r_{2\Lambda'}(\lambda) P_{2\Lambda'},$$

here $P_{2\Lambda'}$ is the projector on $S(2\Lambda')$ in the Clebsch–Gordan decomposition.

By construction, $[H_{n,n+1}, \Delta(a)] = 0\ \forall a$
$+$ coassociativity $\Rightarrow H_{n,n+1} =$ commutes with $\mathcal{E}$, $\mathcal{F}$, $\mathcal{H}$
$\Rightarrow H$ is $U_q(su_2)$–invariant (if the chain is open, i.e. there is no $H_{N,1}$ term)

Known regular $U_q(su_2)$–invariant R–matrices.

"Principal series", $q = 1$ only:

$$R(\lambda) = (1-\lambda)^{-1}\Big(\mathbb{E} - \lambda\,\mathbb{P}\Big) \qquad [Yang, 67]$$

$$R(\lambda) = (1-\lambda)^{-1}\Big(\mathbb{E} - \lambda\mathbb{P} + \frac{\beta\lambda}{\lambda-\alpha}P_0\Big) \qquad [Zamolodchikovs, 79]$$

$$\mathbb{P} = \sum_{\Lambda'=0}^{\Lambda}(-1)^{\Lambda-\Lambda'}P_{2\Lambda'}$$

"Principal series", $q = 1$ and generic q:

$$R(\lambda) = \mathbb{E} + (b^2+1)\frac{1-e^\lambda}{e^\lambda - b^2}P_0 \qquad [Baxter, 82]$$

$$R(\lambda) = \sum_{\Lambda'=0}^{\Lambda}\Big(\prod_{k=\Lambda'+1}^{\Lambda}\frac{[k+\lambda]_q}{[k-\lambda]_q}\Big)P_{2\Lambda'} \qquad \begin{bmatrix} q=1\text{: } Kulish, Reshetikhin, Sklyanin, 81 \\ q \neq 1 : Jimbo, 86 \end{bmatrix}$$

$$[k]_q \equiv \frac{q^k - q^{-k}}{q-q^{-1}}$$

Exceptional solutions:

a) $q = 1$, $\Lambda = 6$:

$$R(\lambda) = P_6 + \frac{1+\lambda}{1-\lambda} P_5 + P_4 + \frac{4+\lambda}{4-\lambda} P_3 + P_2 + \frac{1+\lambda}{1-\lambda} P_1 + \frac{1+\lambda}{1-\lambda}\frac{6+\lambda}{6-\lambda} P_0 .$$

b) $q \neq 1$, $\Lambda = 2$: (Izergin–Korepin, 82):

$$R(\lambda) = P_2 + \frac{[2+\lambda]_q}{[2-\lambda]_q} P_1 + \frac{\{3+\lambda\}_q}{\{3-\lambda\}_q} P_0 ,$$

$\{t\}_q \equiv q^t + q^{-t}$.

Proposition: this list is complete.

AB: St.Petersburg Math. J. 17 (2006), J. Math. Sciences 143 (2007), J. Math. Sciences 143 (2010), in preparation ...

Method: reduction of the Yang–Baxter equation on the subspace spanned by highest weight vectors of weight $(3\Lambda - 2n)$:

$$W^{(n)} = \{ \, v \in S(\Lambda)^{\otimes 3} \;\big|\; \mathcal{E}_{123}\, v = 0 \,, \quad \mathcal{H}_{123}\, v = (3\Lambda - 2n) v \, \}$$

For $R(\lambda) = \sum\limits_{\Lambda'=0}^{\Lambda} r_{2\Lambda'}(\lambda) P_{2\Lambda'}$, the restriction of the Yang–Baxter equation to $W^{(n)}$ reads:

$$\begin{aligned} & A^{(n)}\, D^{(n)}(\lambda - \mu)\, A^{(n)}\, D^{(n)}(\lambda)\, A^{(n)}\, D^{(n)}(\mu) \\ & \quad - D^{(n)}(\mu)\, A^{(s,n)}\, D^{(n)}(\lambda)\, A^{(n)}\, D^{(n)}(\lambda - \mu)\, A^{(n)} = 0, \end{aligned}$$

where $D^{(n)}(\lambda) = \mathrm{diag}\{ r_{2\Lambda}(\lambda), \ldots, r_{2\Lambda - 2n}(\lambda) \}$.

$$A^{(n)}_{kk'} = (-1)^{\Lambda-n}\sqrt{[2\Lambda-2k+1]_q[2\Lambda-2k'+1]_q}\left\{\begin{matrix}\Lambda & \Lambda & \Lambda{-}k\\ \Lambda & 3\Lambda{-}2n & \Lambda{-}k'\end{matrix}\right\}_q.$$

$A^{(n)}$ is orthogonal and symmetric:

$$A^{(n)} = \left(A^{(n)}\right)^t = \left(A^{(n)}\right)^{-1}.$$

$R(\lambda)$ from Temperley–Lieb algebra

Temperley–Lieb algebra TL_N: an associative algebra with generators T_n, $n = 1, \ldots, N$ and relations:

$$\begin{aligned} T_n\, T_m &= T_m\, T_n && \text{for } |n-m| > 1 \\ T_n\, T_{n\pm1}\, T_n &= T_n && \\ T_n\, T_n &= (Q + Q^{-1})\; T_n && Q \in \mathbb{R} \end{aligned}$$

"Baxterization":

$$R_{12}(\lambda) = I + Q\,\frac{1-e^{\lambda}}{e^{\lambda}-Q^2}\; T_{12}, \qquad R_{23}(\lambda) = I + Q\,\frac{1-e^{\lambda}}{e^{\lambda}-Q^2}\; T_{23}$$

solve the Yang–Baxter equation for $Q \neq 1$.

The Temperley–Lieb chain:

$$H = \sum_{n=1}^{N} T_n$$

Spectrum is determined by the minimal polynomial $\mathcal{P}(t)$ such that $\mathcal{P}(H) = 0$.

$$\begin{aligned}
P_1(t) &= t\big(t - [2]_Q\big) \\
P_2(t) &= t\big(t + 1 - [2]_Q\big)\big(t - 1 - [2]_Q\big) \\
P_3(t) &= t\big(t - [2]_Q\big)\big(t^2 - 2[2]_Q t + Q^2 + Q^{-2}\big)\big(t^2 - 3[2]_Q t + 2[3]_Q\big)
\end{aligned}$$

$$\deg P_N(t) = \binom{N+1}{\lfloor \frac{N+1}{2} \rfloor}$$

Baxter's $R(\lambda)$ is a particular case with $T = [\Lambda + 1]_q \, P_0$.

$q > 1$, $\Lambda = 1$ is a deformation of the Heisenberg chain called the XXZ spin chain:

$$H_{XXZ} = \sum_n \sigma_n^1 \, \sigma_{n+1}^2 + \sigma_n^2 \, \sigma_{n+1}^1 + \frac{q + q^{-1}}{2} \, \sigma_n^3 \, \sigma_{n+1}^3 + \frac{q - q^{-1}}{2} \, (\sigma_n^3 - \sigma_{n+1}^3)$$

Spectrum of H_{XXZ} is found via the algebraic Bethe ansatz (with proper boundary conditions).

Let S be a finite dimensional vector space with a scalar product $(\ ,\ )$ and an orthonormal basis $\{e_k\}$. Consider a vector (of unit length)

$$v = \sum_{k,m} V_{km} e_k \otimes e_m \quad \in S \otimes S$$

The orthogonal projector on v is

$$P = v(v,\ ) = \sum_{k,m,k',m'} V_{km}\, \bar{V}_{k'm'}\, e_k \otimes e_m \otimes e^{k'} \otimes e^{m'}\ ,$$

where $\{e^k\}$ is the orthonormal basis of the dual space $S^* \sim S$, i.e. $e^k(e_m) = \delta_{km}$.

Temperley–Lieb relation:

$$P_{12}P_{23}P_{12} = \mu\, P_{12} \quad \Leftrightarrow \quad V\bar{V} \sim \text{unitary}$$

$U_q(su_2)$: S is an irreducible module of weight Λ.
Basis: weight vectors, i.e. $H\, e_k = (\Lambda + 1 - 2k)\, e_k$, $k = 1, \ldots, \Lambda + 1$.

Consider a weight vector $v \in S(\Lambda') \subset S(\Lambda) \otimes S(\Lambda)$. V is a matrix of its Clebsch–Gordan coefficients.
Note: $\Delta(H)v = 0 \Rightarrow V$ is anti–diagonal matrix $\Rightarrow V\bar{V}$ is diagonal (for $q > 1$, $\bar{V} = V$).

Proposition. The Temperley–Lieb relations holds for the projector on $v = |\Lambda', 0\rangle$ for
a) $\Lambda' = 0$, any Λ;
b) $\Lambda' = 1$, $\Lambda = 1$;
c) $\Lambda' = 4$, $\Lambda = 3$, only if $q = 1$.

a) yields Temperley–Lieb chain with $Q = q^{\Lambda}$.
b) yields $Q = q$ and

$$H_{n,n+1} = \sigma_n^1\, \sigma_{n+1}^2 + \sigma_n^2\, \sigma_{n+1}^1 - \frac{q + q^{-1}}{2}\, \sigma_n^3\, \sigma_{n+1}^3 + \frac{q - q^{-1}}{2}\, (\sigma_n^3 - \sigma_{n+1}^3)$$

c) yields $Q + Q^{-1} = 4$.

Thank you!