SLE, KPZ &
Liouville Quantum Gravity

Bertrand Duplantier

Institut de Physique Théorique, Saclay, France

& Scott Sheffield, MIT Math

International Conference "Random Processes, Conformal Field Theory & Integrable Systems"

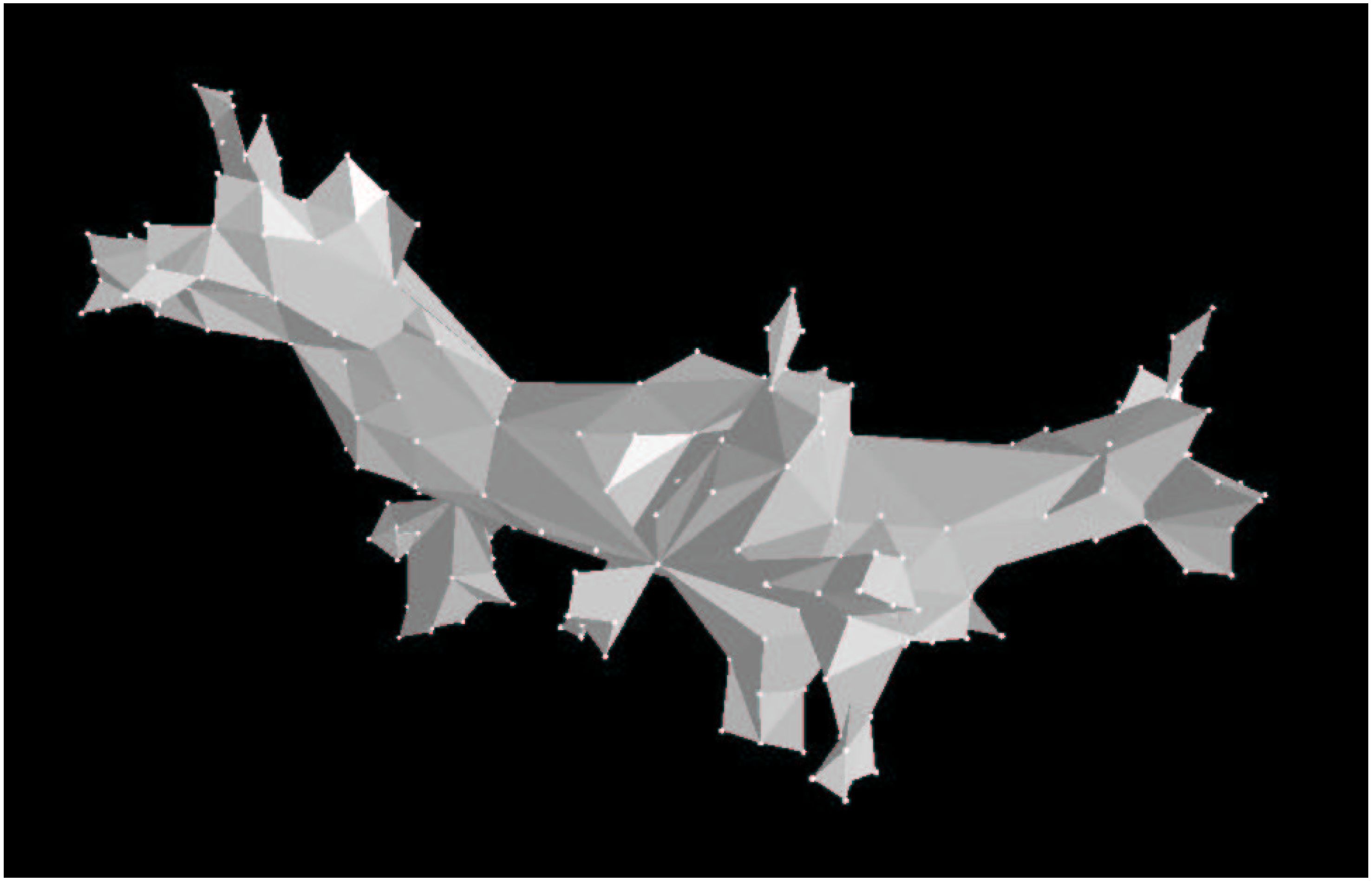
Independent University of Moscow

Poncelet Laboratory

Moscow, Russia

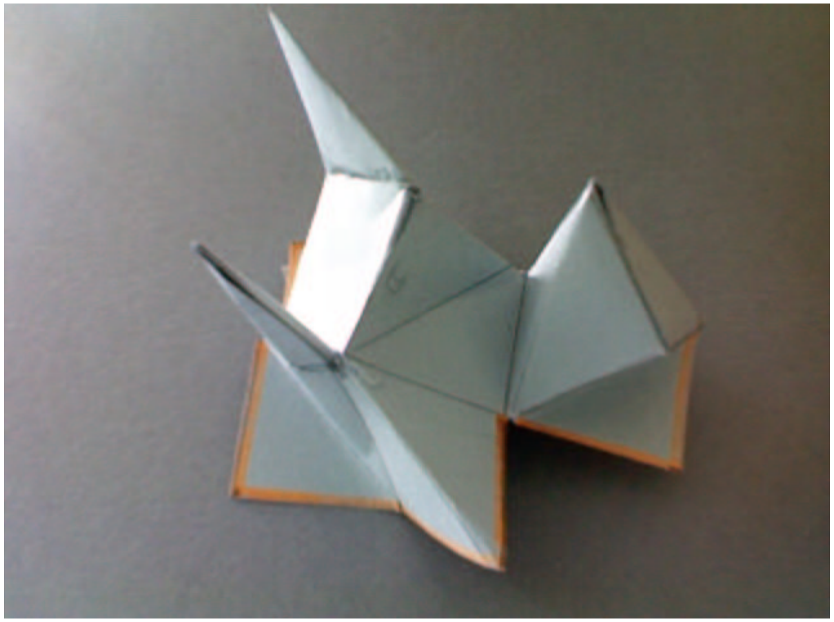
September 19 – 23, 2011

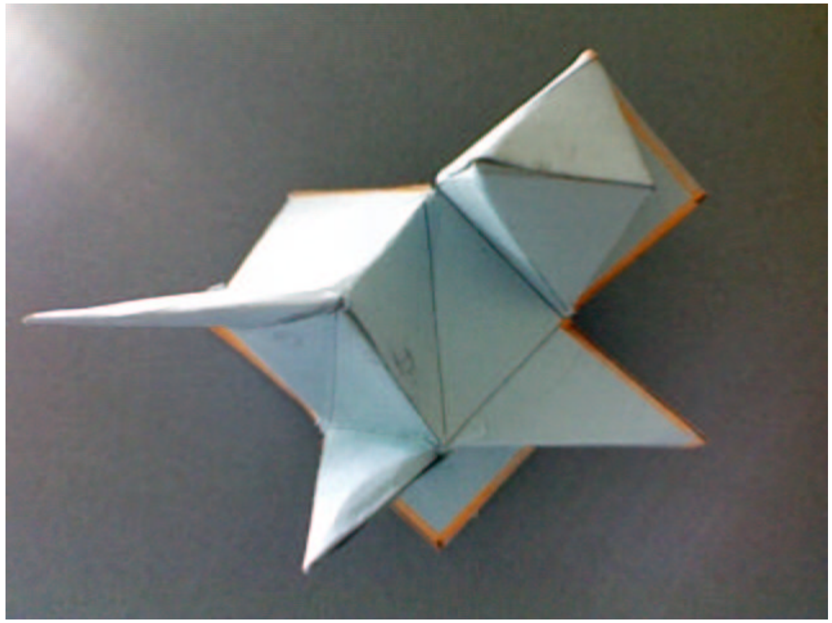
A Random Surface



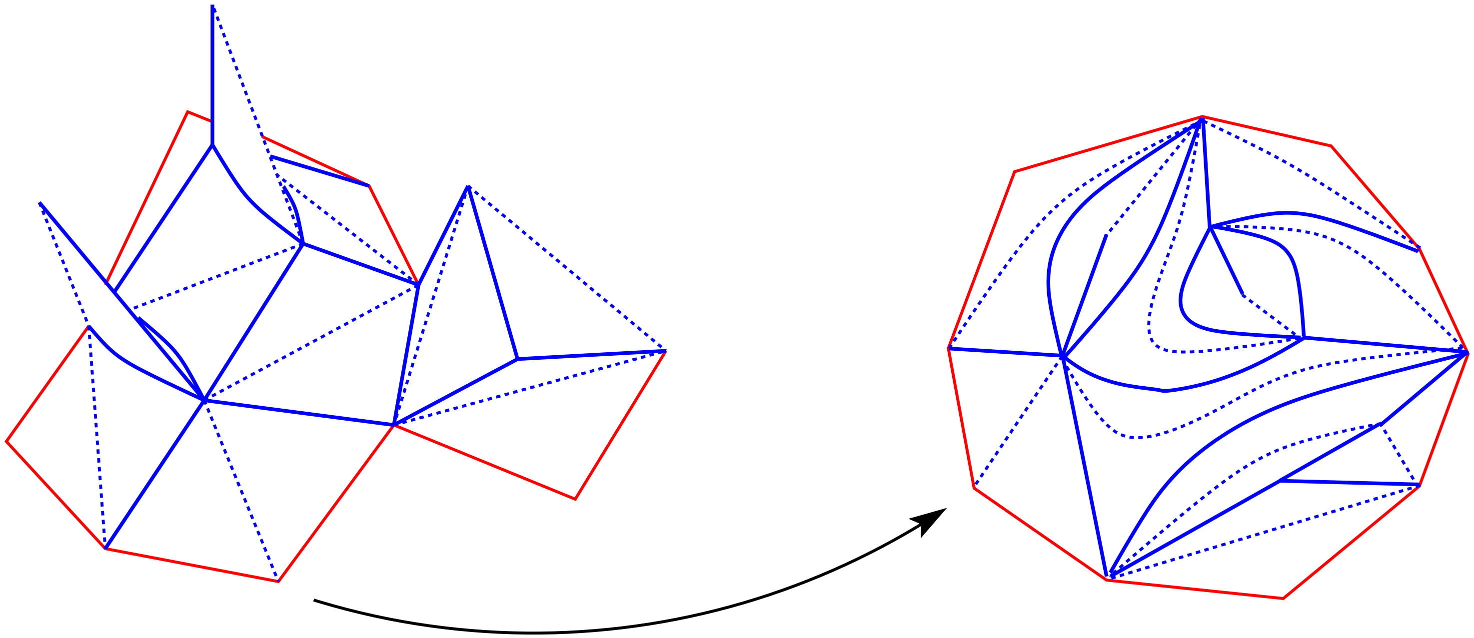
[Courtesy of G. Chapuy (2009)]

A Random Quadrangulation





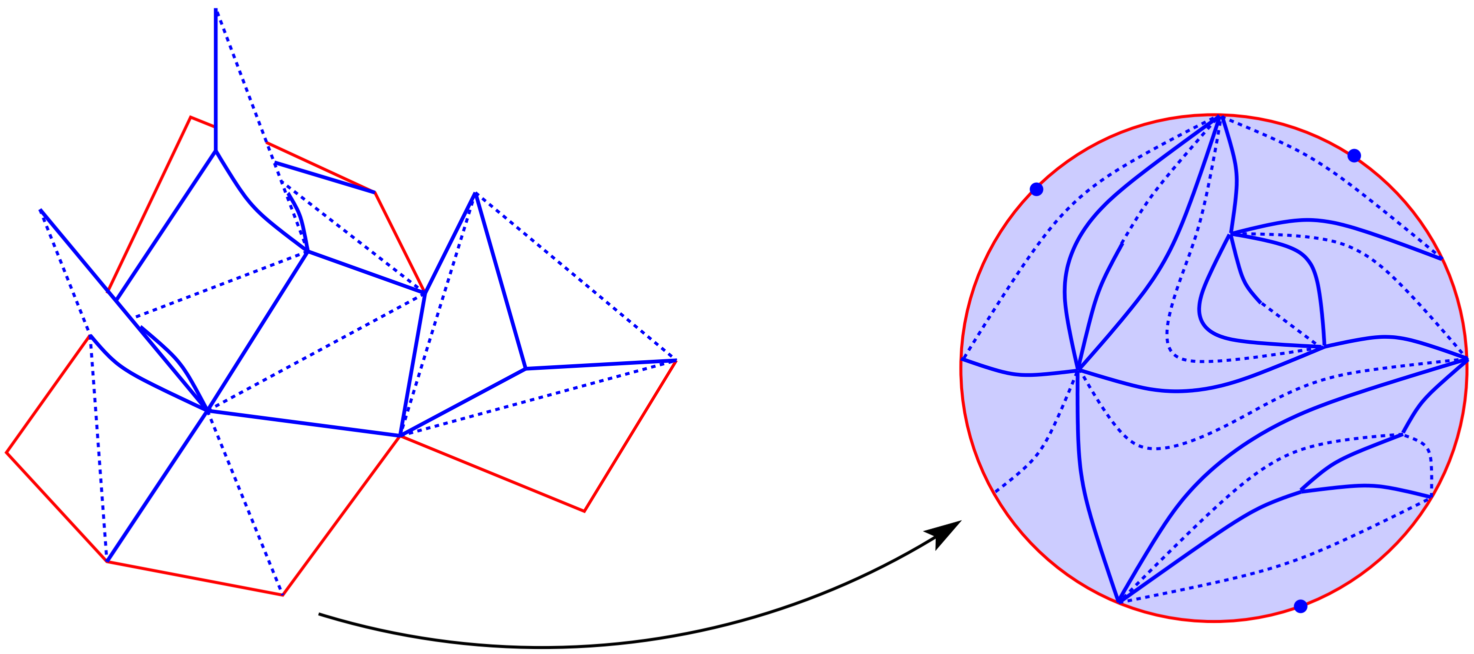
Random Quadrangulation & Random Planar Map



RS & Random Matrices *BIPZ '78; Ambjørn, Durhuus, Fröhlich, Jonsson '83-85; David '85; Boulatov, Kazakov, Kostov, Migdal '85...*

Bijective Combinatorics *Cori, Vauquelin '81; Schaeffer '97; Angel, Schramm '03; Bouttier, Di Francesco, Guitter '04; Le Gall, Miermont...*

Random Quadrangulation & Conformal Map to $\mathbb{D}$

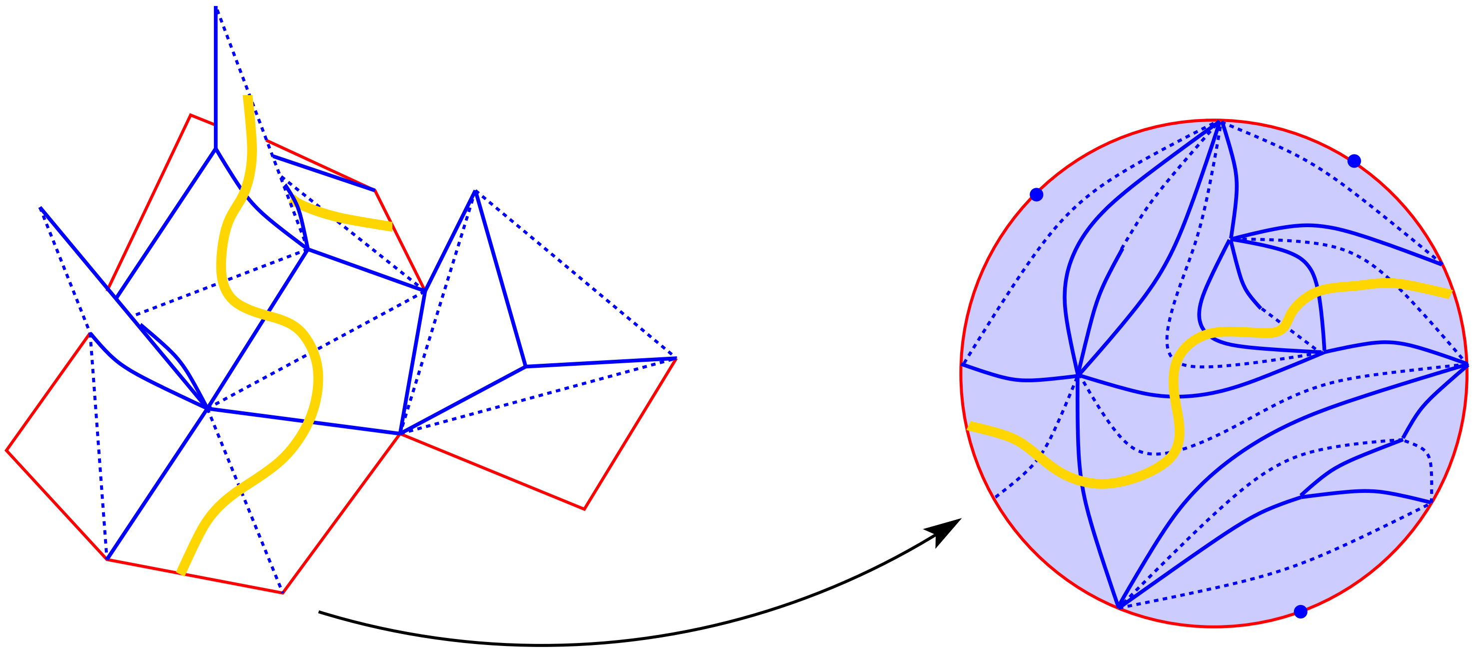


In the continuum scaling limit: **Liouville Quantum Gravity**

A.M. Polyakov ’81

Correlation Functions *Seiberg, ’90; Goulian, Li ’91; Ginsparg, Moore ’93; Dorn, Otto ’94; Takhtajan ’95; Teschner ’95; Zamolodchikov^2 ’96; Fateev-ZZ ’00; Ponsot, Teschner ’02; Kostov, Ponsot, Serban ’04...*

Random Quadrangulation & Random Sets & Paths



Ising, SAW, $O(N)$ & Potts models: **Random Matrix Models**

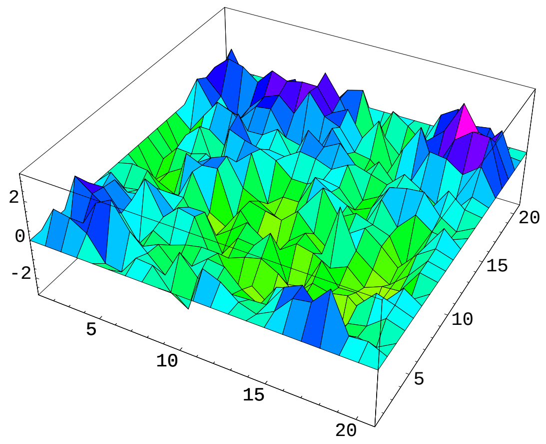
Kazakov '86; D. & Kostov '88; Kostov; Daul; Eynard, Zinn-Justin^2...

Bijective Combinatorics *Chassaing & Schaeffer '02;*

Bousquet-Mélou & Schaeffer '02; BDFG '02; Bernardi & B.-M. '09...

Continuum: **Liouville Gravity & Conformal Field Theory**

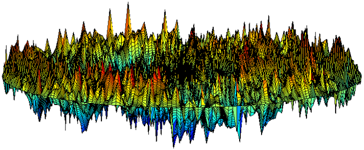
Gaussian Free Field (GFF)



Distribution h with *Gaussian weight* exp $\left[-\frac{1}{2}(h,h)_\nabla\right]$, and **Dirichlet inner product** in domain D

$$(f_1, f_2)_\nabla := (2\pi)^{-1} \int_D \nabla f_1(z) \cdot \nabla f_2(z) d^2z$$
$$= \mathrm{Cov}\big((h, f_1)_\nabla, (h, f_2)_\nabla\big)$$

◊ STARRING THE GFF! *(Courtesy of N.-G. Kang)* ◊



Liouville QG

Random Measure

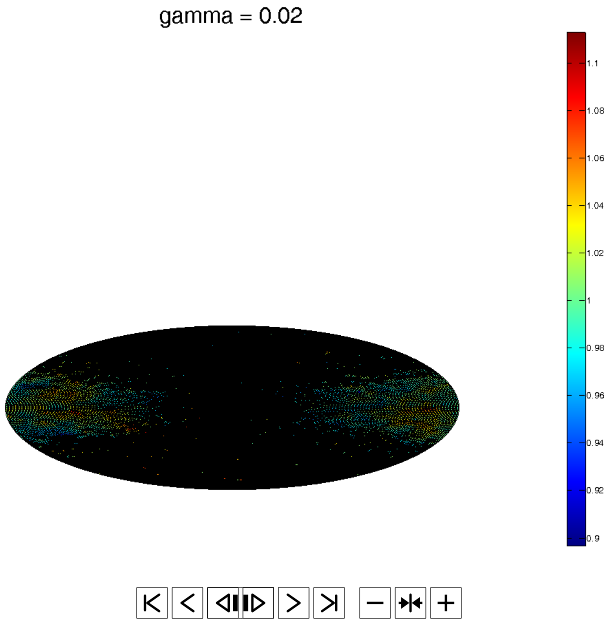
$$d\mu = \text{“}e^{\gamma h} d^2 z\text{”}$$

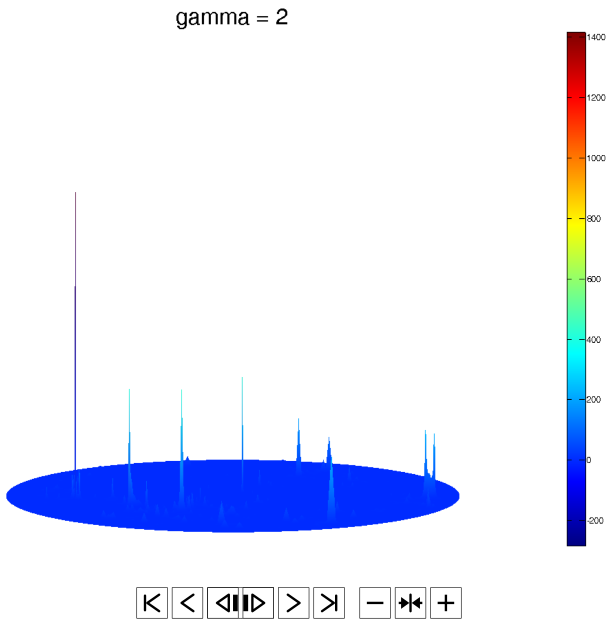
◊

The Emergence of Quantum Gravity

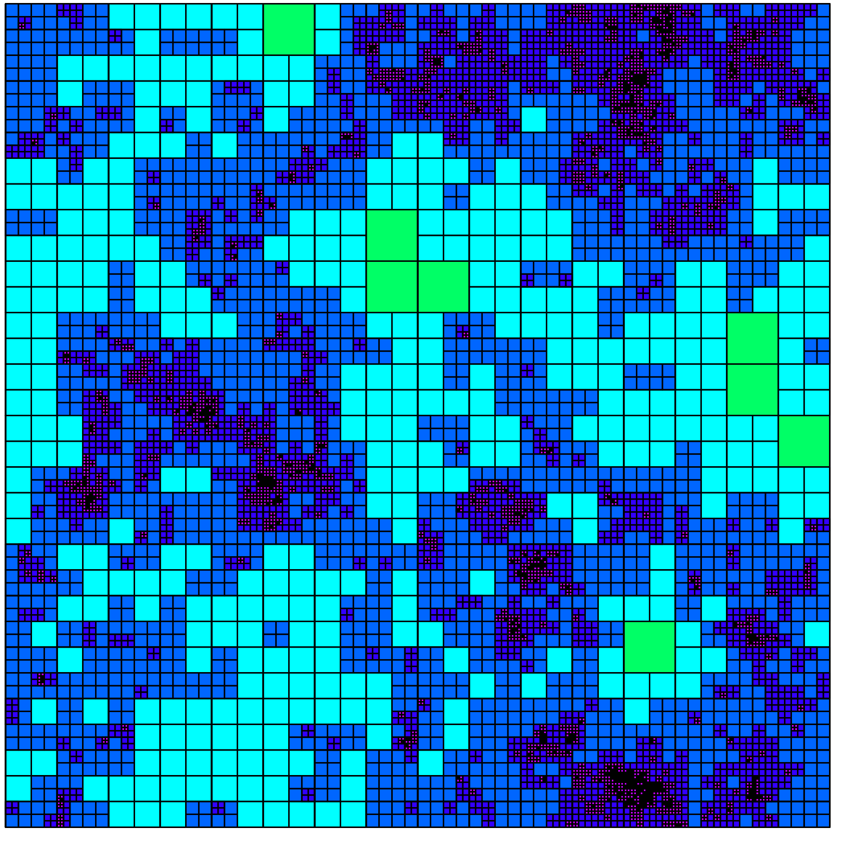
(Courtesy of N.-G. Kang)

◊


gamma = 0.02
1.1
1.08
1.06
1.04
1.02
1
0.98
0.96
0.94
0.92
0.9


gamma = 2
1400
1200
1000
800
600
400
200
0
-200

Discrete Quantum Gravity Measure ($\gamma = 3/2$)



Euclidean squares of similar quantum area δ

- Regularization

$h_\varepsilon(z)$ **mean value of** h **on circle** $\partial B_\varepsilon(z)$

- Variance

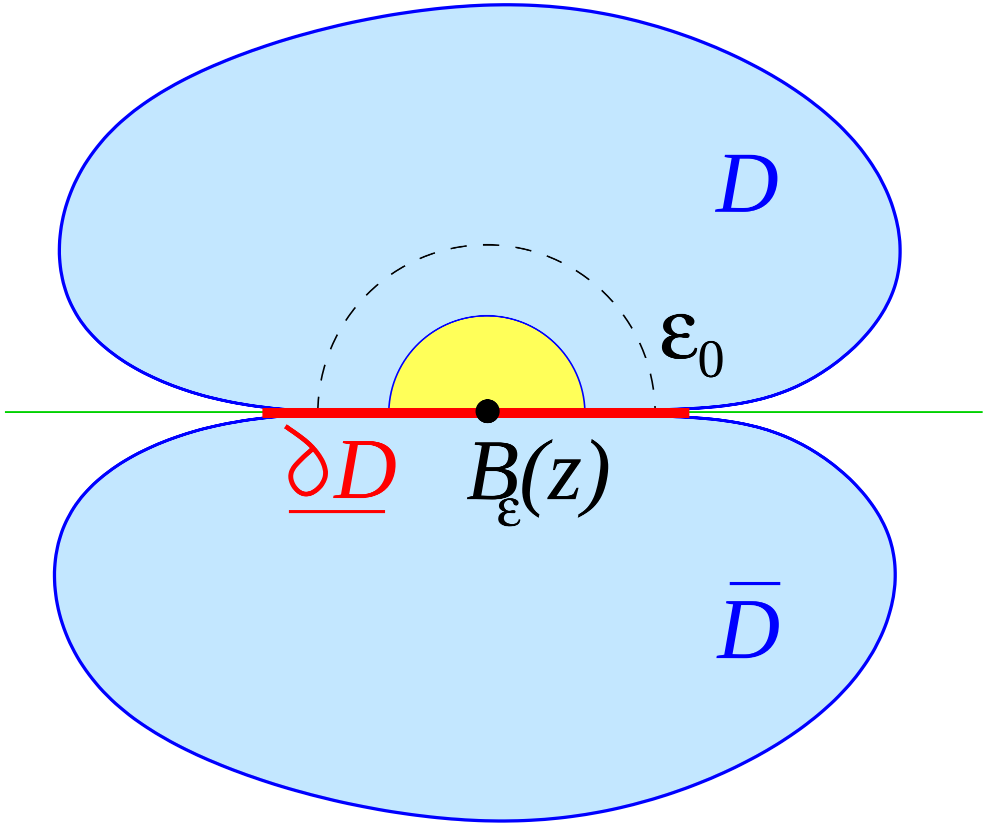
$$\operatorname{Var} h_\varepsilon(z) = \log[C(z,D)/\varepsilon]$$

$C(z,D)$ *conformal radius of* D *viewed from* z

$h_\varepsilon(z)$ **Gaussian random variable**

$$\mathbb{E}\, e^{\gamma h_\varepsilon(z)} = e^{\gamma^2 \operatorname{Var} h_\varepsilon(z)/2} = \left(\frac{C(z,D)}{\varepsilon}\right)^{\gamma^2/2} \qquad \square$$

Boundary Liouville Quantum Gravity



- *GFF with free boundary conditions on* $\underline{\partial D}$*;*
- *Half-circle averages* $\hat{h}_\varepsilon(z)$.

Quantum Area Measure

$$d\mu_\varepsilon := \exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2} d^2 z$$

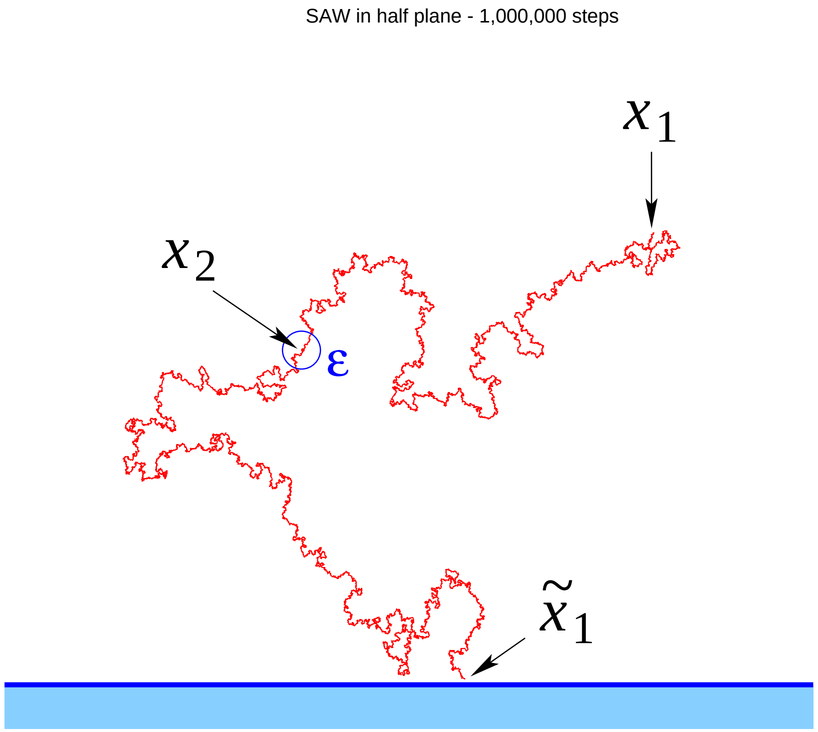
converges to a random measure as $\varepsilon \to 0$ *for* $\gamma < 2$.

Quantum Boundary Measure

$$d\hat{\mu}_\varepsilon := \exp\left[\frac{\gamma}{2} \hat{h}_\varepsilon(z)\right] \varepsilon^{\gamma^2/4} dz$$

converges to a boundary random measure as $\varepsilon \to 0$ *for* $\gamma < 2$.

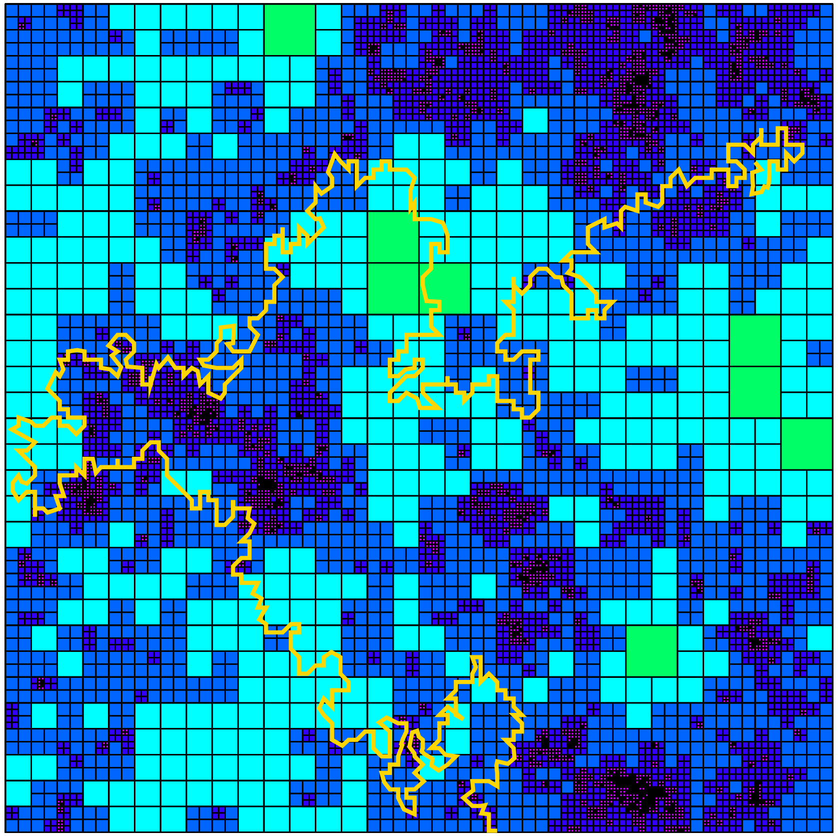
Scaling Exponents of (Random) Fractals in $\mathbb{H}$



Probabilities & Hausdorff Dimensions (e.g., SLE_κ)

$\mathbb{P} \asymp \varepsilon^{2x}, \quad \tilde{\mathbb{P}} \asymp \varepsilon^{\tilde{x}}, \qquad D = 2 - 2x_2 \qquad (= 1 + \kappa/8)$

Quantum Gravity Scaling Exponents



$$\mathbb{P} \asymp \delta^{\Delta}, \qquad \tilde{\mathbb{P}} \asymp \tilde{\delta}^{\tilde{\Delta}}$$

KPZ ’88

x and Δ ($\tilde{x}$ and $\tilde{\Delta}$) are related by the **KPZ formula**

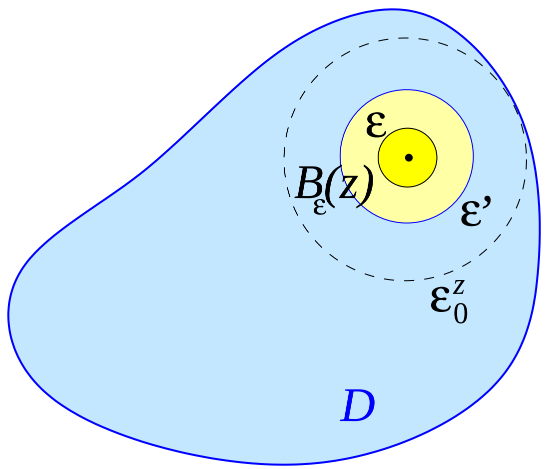
$$x = \left(1 - \frac{\gamma^2}{4}\right)\Delta + \frac{\gamma^2}{4}\Delta^2$$

KPZ is a Theorem *[D. & Sheffield, ’08]*

PRL **102**, *150603 (2009) & Invent. Math.* **185**, *333 (2011)*

Kazakov ’86; D. & Kostov ’88 [Random matrices]
David; Distler & Kawai ’88 [Liouville field theory]
Benjamini & Schramm ’08; Rhodes & Vargas ’11 [Math]
David & Bauer ’09

GFF & Brownian Motion



- $h_\varepsilon(z)$ mean value of h on circle $\partial B_\varepsilon(z)$

- Define $t := -\log \varepsilon$, $\mathcal{B}_t := h_{\varepsilon = e^{-t}}(z)$; for z fixed, the law of $\mathcal{B}_t$ is **standard Brownian motion** in t

$$\mathrm{Var}\left[(h_\varepsilon - h_{\varepsilon'})(z)\right] = |\log(\varepsilon/\varepsilon')| = |t - t'| = \mathrm{Var}\left[\mathcal{B}_t - \mathcal{B}_{t'}\right] \quad \square$$

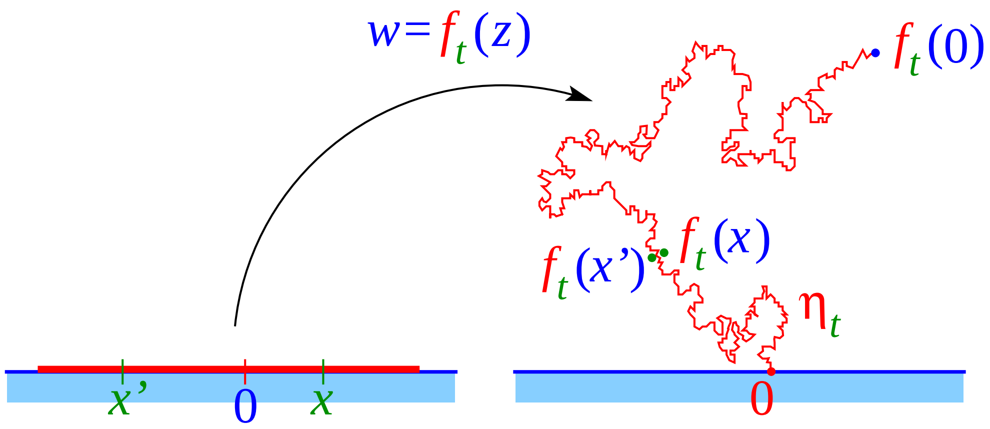
SLE - GFF (QG) Coupling

(Dubédat, 2009)

Sheffield, arXiv:1012.4797

D. & Sheffield, arXiv:1012.4800, PRL *(2011)*

“*Zipping-up*” SLE Map



Let f_t be the (reverse) SLE$_\kappa$ conformal map

$$z \in \mathbb{H} \to w = f_t(z) \in \mathbb{H} \setminus \eta_t,$$

with trace η_t and tip $f_t(0)$ [$t = 0,\ f_0(z) = z$].
It satisfies the stochastic differential equation (B_t *standard Brownian motion*)

$$d f_t(z) = -2dt / f_t(z) - \sqrt{\kappa}\, dB_t.$$

SLE Martingale

Real stochastic process in the upper-half plane:

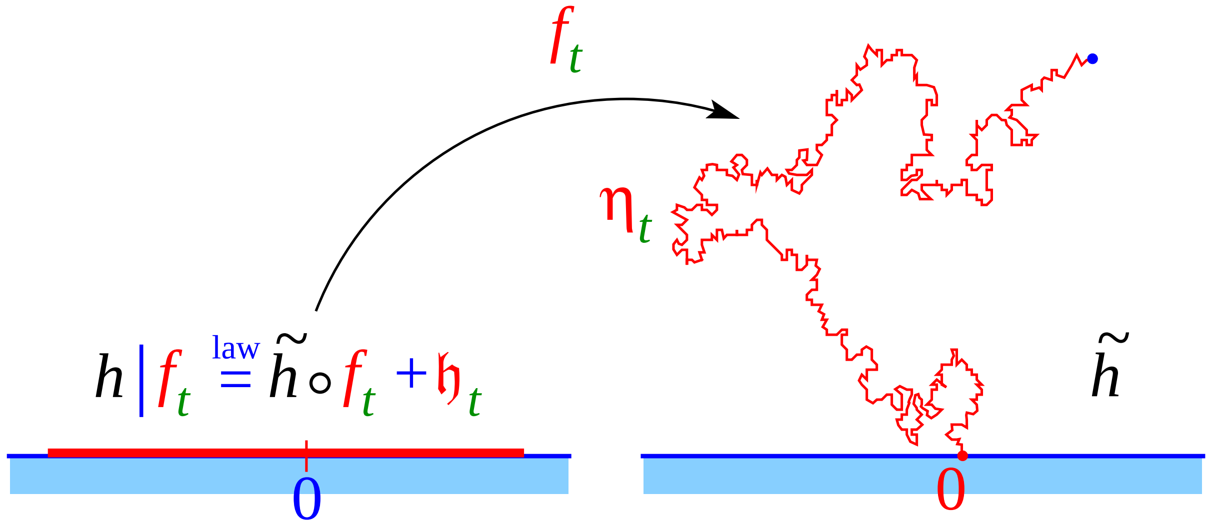
$$\begin{aligned} \mathfrak{h}_0(z) &:= \frac{2}{\sqrt{\kappa}} \log|z|, \\ \mathfrak{h}_t(z) &:= \mathfrak{h}_0 \circ f_t(z) + Q \log|f_t'(z)|. \end{aligned}$$

This process $\mathfrak{h}_t(z)$ is a *martingale* (so that $\mathbb{E}\mathfrak{h}_t(z) = \mathfrak{h}_0(z)$) for the particular choice:

$$Q = \sqrt{\kappa}/2 + 2/\sqrt{\kappa},$$

for which $d\mathfrak{h}_t(z) = -\Re[2/f_t(z)]dB_t$.

SLE & GFF Coupling

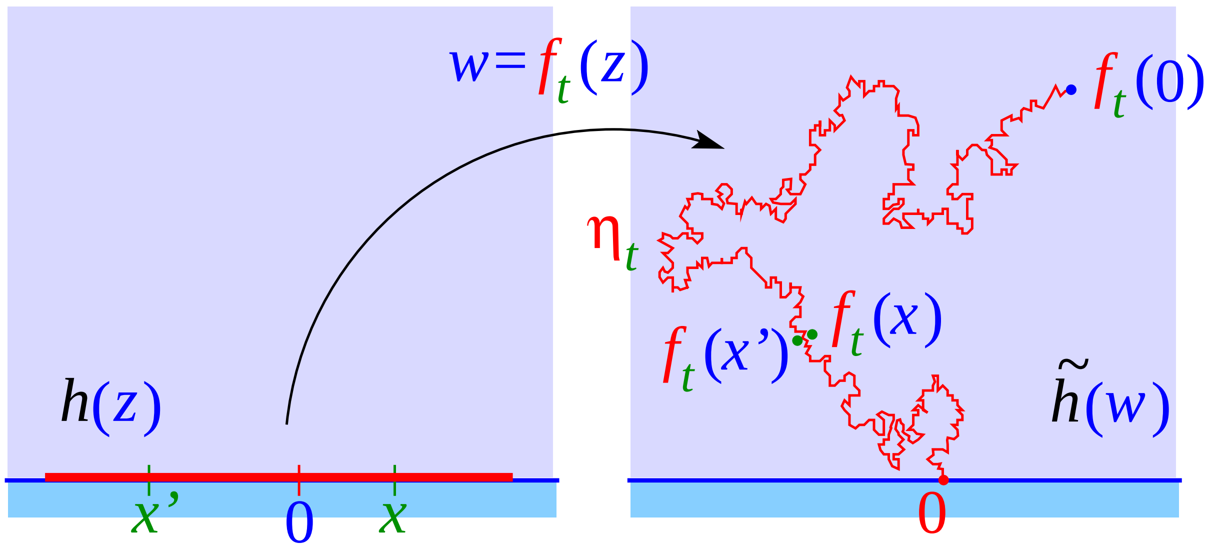


Consider $h := \tilde{h} + \mathfrak{h}_0$, sum of the GFF $\tilde{h}$ on $\mathbb{H}$ with *free boundary conditions* on $\mathbb{R}$, and of the deterministic function $\mathfrak{h}_0$. Given f_t, the conditional law of h is

$$h(z)|\, f_t \overset{\text{(law)}}{=} \tilde{h}\circ f_t(z) + \mathfrak{h}_t(z),$$

where $\tilde{h}\circ f_t$ is the pullback of the free boundary GFF $\tilde{h}$.

Conformal Welding



$$h\,|\,f_t \overset{\text{law}}{=} \tilde{h} \circ f_t + \mathfrak{h}_t$$

Conformal welding: the *quantum boundary lengths* of any pair of real segments $[0,x]$ and $[x',0]$ such that $f_t(x)=f_t(x')$ on the SLE trace are *equal* for $h=\tilde{h}+\mathfrak{h}_0$ *[Sheffield, 2010].*

SLE Exponential Martingales & KPZ Relation

$$\begin{aligned} \mathcal{M}_t^{\alpha}(z) &:= \mathbb{E}\left(e^{\alpha h(z)} \,|\, f_t\right), \;\; \alpha \in \mathbb{R} \\ \left(e^{\alpha h(z)} \,|\, f_t\right) d^2 z &\overset{(\text{law})}{=} \left|f_t'(z)\right|^{d-2} e^{\alpha h(w)} d^2 w \\ d &:= \alpha Q - \alpha^2/2 \;\; (\text{KPZ}) \end{aligned}$$

where $w = f_t(z)$, $d^2 w = |f_t'(z)|^2 d^2 z$.

Liouville Quantum Measure

$$e^{\gamma h(z)}|f_t\, d^2z \overset{(\text{law})}{=} e^{\gamma h(w)}\, d^2w \quad \text{(conformal invariance)}$$

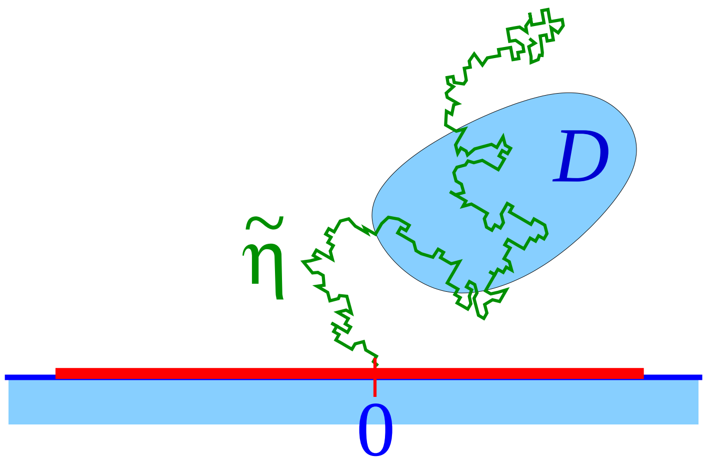
for $d = 2 = \gamma Q - \gamma^2/2$, *i.e.*, $Q = \gamma/2 + 2/\gamma = \sqrt{\kappa}/2 + 2/\sqrt{\kappa}$

$$\gamma = \sqrt{\kappa \wedge 16/\kappa},\ \ \gamma' = 4/\gamma$$

- $\gamma \leqslant 2$: *KPZ prediction* $\gamma = \left(\sqrt{25-c} - \sqrt{1-c}\right)/\sqrt{6}$ for the *central charge* $c = \frac{1}{4}(6-\kappa)(6-16/\kappa) \leqslant 1$ of the SLE's CFT coupled to gravity.
- $\gamma' = 4/\gamma > 2$: *Duality* property of Liouville quantum gravity; the quantum measure develops atoms with localized area.

Conformally welding two γ-Liouville quantum surfaces produces SLE_κ.

SLE Natural Length



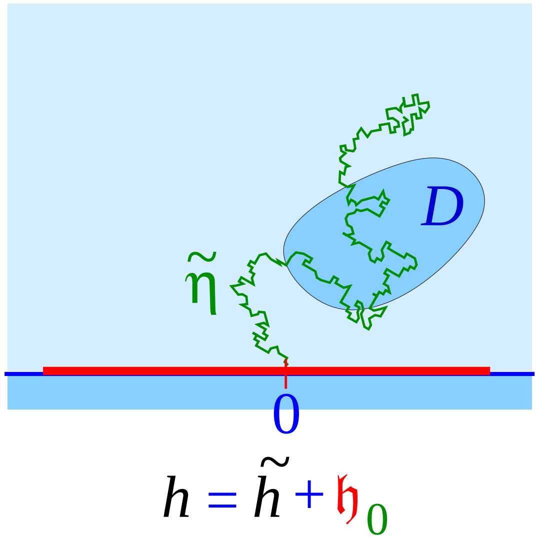
Expected (w.r.t. the $\mathrm{SLE}_{\kappa\in[0,8]}$ law) *length* of an infinite SLE $\tilde{\eta}$ in D *(Lawler & Sheffield, 2009)*

$$\nu(D) = \int_D G(z) d^2 z,$$

SLE Green's function in $\mathbb{H}$:

$$G(z) := |z|^a |\Im z|^b, \; a = 1 - 8/\kappa, \; b = 8/\kappa + \kappa/8 - 2.$$

SLE Quantum Length



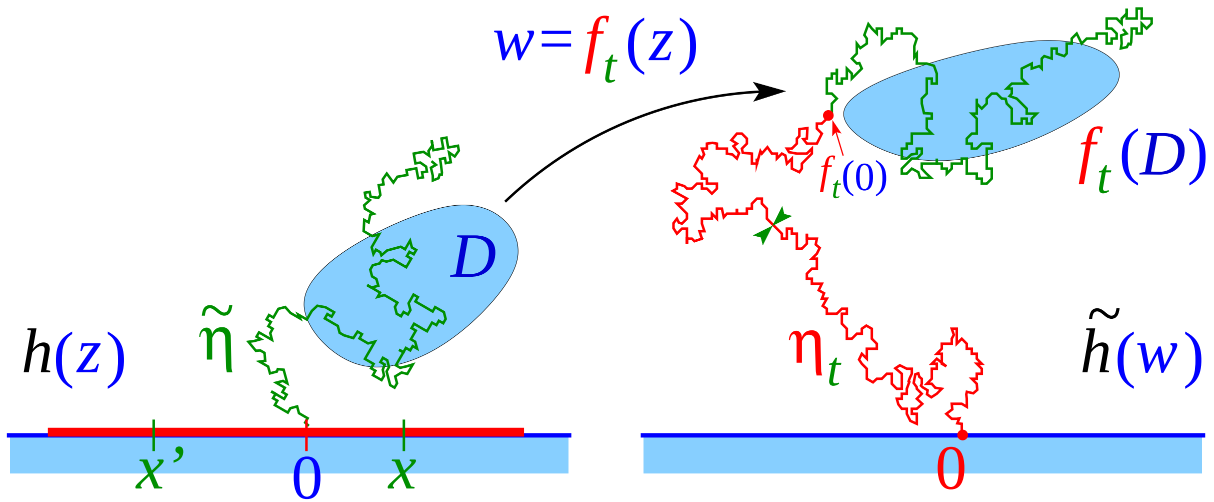
Expected (w.r.t. $\tilde{\eta}$, given h) Liouville *quantum length* ν_Q in D

$$\nu_Q(D,h) := \int_D e^{\alpha h(z)} G(z) d^2 z,$$

$\alpha = \sqrt{\kappa}/2$ ($= \gamma/2$ for $\kappa \leqslant 4$, and $\gamma'/2$ for $\kappa > 4$) satisfies KPZ for the SLE Hausdorff dimension $d = 1 + \kappa/8$.

[Doob Meyer, second moment method.]

Expected SLE Quantum Length



$$h\,|\,f_t \overset{\text{law}}{=} \tilde{h}\circ f_t + \mathfrak{h}_t$$

$$\mathbb{E}[\nu_Q(D,h)\,|\,f_t] = \int_D \mathcal{M}_t^{\alpha}(z)G(z)d^2z$$

$$\mathbb{E}\nu_Q(D,h) = \int_D \mathcal{M}_0^{\alpha}(z)G(z)d^2z = \int_D (\sin\vartheta)^{8/\kappa-2}d^2z,$$

with $\vartheta := \arg z$. It is finite for $\kappa \in [0,8)$ and coincides with the *Euclidean area* of D for $\kappa = 4$.

PERSPECTIVES

- *Scaling limits of discrete models on random planar graphs*
- *Quantum wedges and cones*
- *Quantum bubbles and foam* ($\gamma\gamma' = 4$ *duality*)
- *KPZ & Supersymmetry*
- *Geodesics & random metrics*

$\diamondsuit \quad \diamondsuit \quad \diamondsuit$