# Statistical Models on random lattices

Bertrand Eynard, CERN, IPHT CEA Saclay
Moscow

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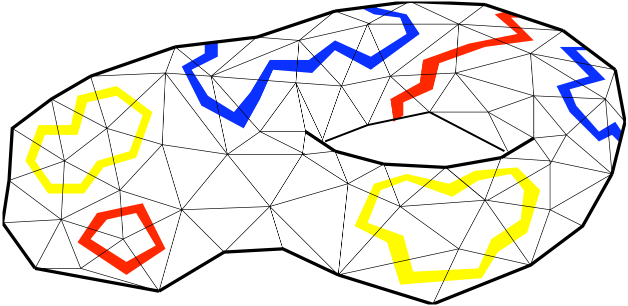


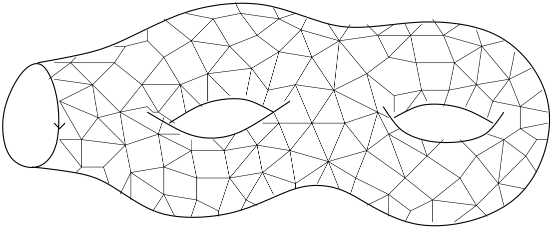
# Contents

# 1. Introduction

Statistical models on random lattices

# Statistical Models

On a random map, of given topology



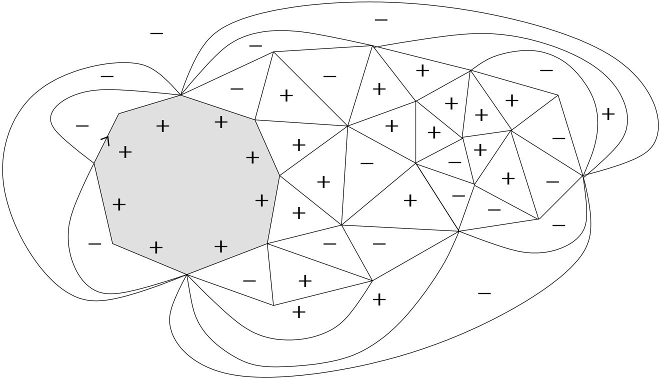
• Genus = $g$

• Number of "boundaries" (boundary = marked face with a marked edge) = $n$.

# Statistical Models

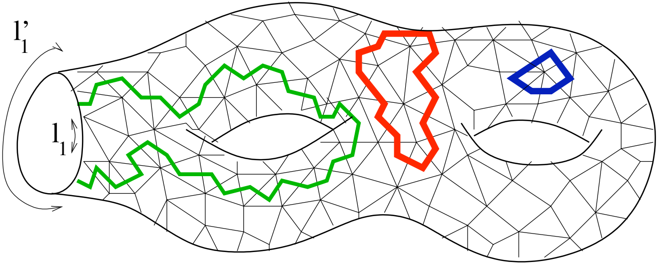
add a statistical model

Example: Ising model.
Map, where each polygon carries a "spin" + or -.



# Statistical Models

Example: $O(\mathfrak{n})$ model.
Map, where $\mathfrak{n}-$colored loops are drawn on triangles




Other models: Potts model, Chain model, 6-vertex model, 3-color model have been solved,
There exist other statistical models which have not been solved...

# Statistical Models

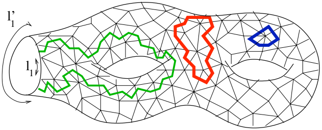
What we can do:

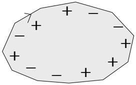
compute the number of configurations (or its generating function), having

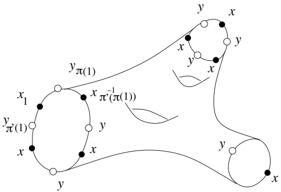
- given topology (given genus and number of marked faces)
- given number of $k$–gons
- given boundary configuration

and depending on the model:

- given total lenght of loops, or total number of $+$ spins, or given number of $+|-$ edges, connectivity pattern of loops ending on boundaries, ...



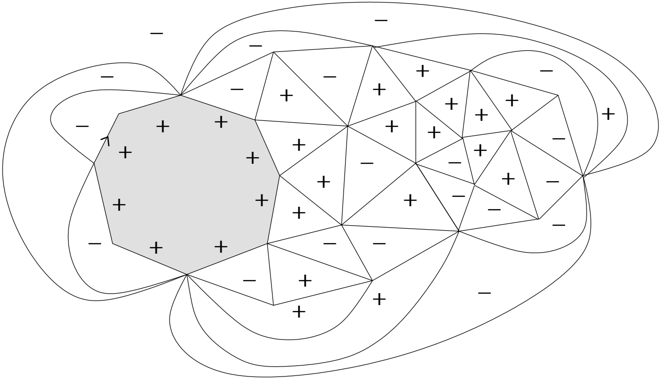


# 2. Ising Model

# Statistical Models

Example: Ising model.

Map, where each polygon carries a "spin" + or -.

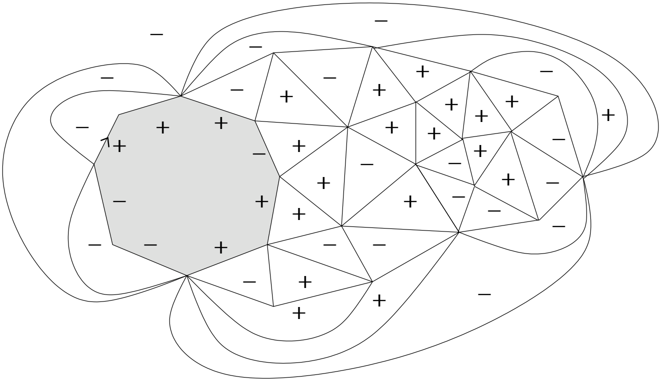


# Statistical Models

Example: Ising model.

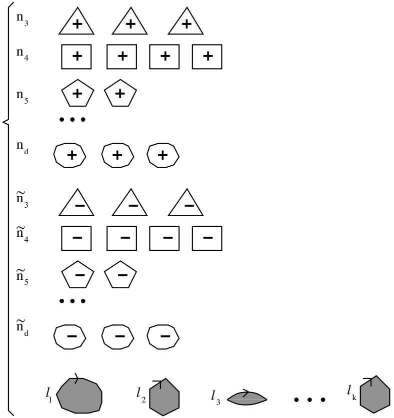
Map, where each polygon carries a "spin" + or -.

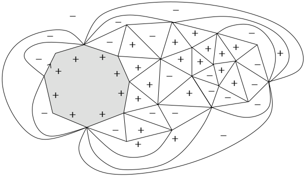
Marked faces can carry spins on their boundaries



# Ising Model

## Rules for constructing an Ising model map



# Ising Model

Generating function for maps having $+$ spins boundaries.

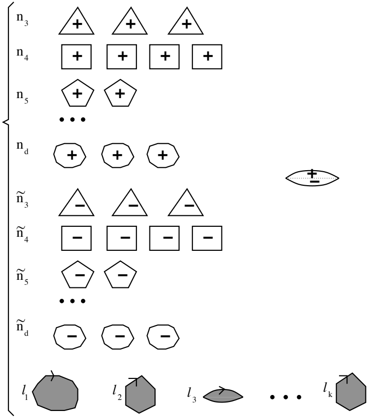
We define:

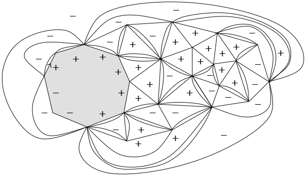
$$W_n^{(g)}(x_1,\dots,x_n;t;t_3,\dots,t_d;\tilde t_3,\dots,\tilde t_{\tilde d};c_{++},c_{+-},c_{--})$$

$$= \sum_v t^v \sum_{S\in\mathbb{M}_{g,n}(v)} \frac{1}{\#\mathrm{Aut}(S)} \frac{t_3^{n_3(S)}\dots t_d^{n_d(S)}\ \ \tilde t_3^{\tilde n_3(S)}\dots \tilde t_{\tilde d}^{\tilde n_{\tilde d}(S)}}{x_1^{1+l_1(S)}\dots x_n^{1+l_n(S)}} (c_{++})^{n_{++}(S)}\,(c_{--})^{n_{--}(S)}\,(c_{+-})^{n_{+-}(S)}$$

$v = \#$ vertices, $n_{\epsilon\epsilon'} = \#$ edges separating spins $\epsilon|\epsilon'$.

## Modified–Rules for constructing an Ising model map



# Ising Model

Rewriting generating function for maps having $+$ spins boundaries.

$$W_n^{(g)}(x_1, \dots, x_n; t; t_3, \dots, t_d; \tilde{t}_3, \dots, \tilde{t}_{\tilde{d}}; c_{++}, c_{+-}, c_{--})$$

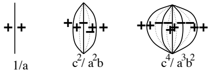
$$= \sum_v t^v \sum_{S \in \mathbb{M}_{g,n}(v)} \frac{1}{\#\mathrm{Aut}(S)} \frac{t_3^{n_3(S)} \dots t_d^{n_d(S)} \;\; \tilde{t}_3^{\tilde{n}_3(S)} \dots \tilde{t}_{\tilde{d}}^{\tilde{n}_{\tilde{d}}(S)}}{x_1^{1+l_1(S)} \dots x_n^{1+l_n(S)}} c^{n_2(S)} \; a^{-n_{++}(S)} \; b^{-n_{--}(S)}$$

where

$$\begin{pmatrix} c_{++} & c_{+-} \\ c_{+-} & c_{--} \end{pmatrix}^{-1} = \begin{pmatrix} a & -c \\ -c & b \end{pmatrix}$$

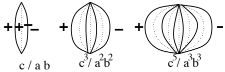
# Ising Model

weight for $+|+$ edges:



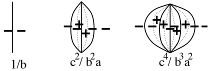

$$c_{++} = \frac{1}{a} + \frac{c^2}{a^2 b} + \frac{c^4}{a^3 b^2} + \cdots = \frac{b}{ab - c^2}$$

weight for $+|-$ edges:




$$c_{+-} = \frac{c}{ab} + \frac{c^3}{a^2 b^2} + \frac{c^5}{a^3 b^3} + \cdots = \frac{c}{ab - c^2}$$

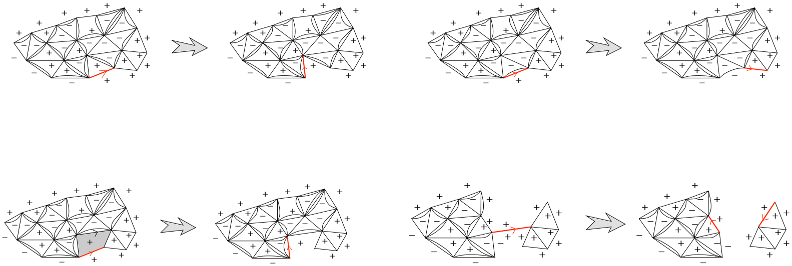
weight for $-|-$ edges:



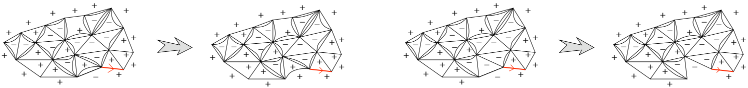

$$c_{--} = \frac{1}{b} + \frac{c^2}{ab^2} + \frac{c^4}{a^2 b^3} + \cdots = \frac{a}{ab - c^2}$$

# Tutte equations

Boundary of type $\overbrace{+,\dots,+}^{l},\overbrace{-,\dots,-}^{k}$, with $0 \leq k \leq \tilde{d}$:



Boundary of type $\overbrace{+,\dots,+}^{l},\overbrace{-}^{1}$:



# Solution of Tutte equations, planar case

**Theorem:** Solution for $W_1^{(0)}$ (planar, 1 marked face):
Let

$$c\,Y(x) \stackrel{\text{def}}{=} ax - \sum_{j=3}^{d} t_j x^{j-1} - W_1^{(0)}(x)$$

It satisfies a "rational" algebraic equation $0 = E(x, Y)$.
More explicitly, parametric solution $x = x(z)$, $Y(x) = y(z)$ given by:

$$\begin{cases} x(z) = \gamma z + \sum_{j=0}^{\tilde{d}-1} \alpha_j\, z^{-j} \\ y(z) = \gamma z^{-1} + \sum_{j=0}^{d-1} \beta_j\, z^j \end{cases}$$

where $\gamma$, $\alpha_j$, $\beta_j$ are the unique solution of

$$\begin{cases} ax(z) - \sum_{j=3}^{d} t_j x(z)^{j-1} = c\,y(z) + \frac{t}{\gamma} z^{-1} + O(z^{-2}) \\ by(z) - \sum_{j=3}^{\tilde{d}} \tilde{t}_j y(z)^{j-1} = c\,x(z) + \frac{t}{\gamma}\, z + O(z^2) \end{cases}$$

# Solution of Tutte equations, planar, 2 boundaries

We define redefine the generating functions as functions of the variable $z$:

$$\omega_n^{(g)}(z_1,\dots,z_n) = W_n^{(g)}(x(z_1),\dots,x(z_n))\,x'(z_1)\dots x'(z_n) + \frac{\delta_{n,2}\delta_{g,0}\,x'(z_1)\,x'(z_2)}{(x(z_1)-x(z_2))^2}$$

**Theorem:** [Kazakov& al ~90's] the 2-point function is universal:

$$\omega_2^{(0)}(z_1,z_2) = \frac{1}{(z_1-z_2)^2}$$

# Solution of Tutte equations, all topologies

**Theorem:** [Chekhov-E-Orantin 05,06] All other "stable" topologies (i.e. $2g-2+n>0$) are given by the "Topological recursion":

$$\omega^{(g)}_{n+1}(z_0,z_1,\dots,z_n) = \sum_i \mathop{\mathrm{Res}}_{z\to a_i} K(z_0,z)\Big[\omega^{(g-1)}_{n+2}(z,\zeta(z),z_1,\dots,z_n) + \sum_h \sum'_{I\uplus \bar I=\{z_1,\dots,z_n\}} \omega^{(h)}_{1+\#I}(z,I)\,\omega^{(g-h)}_{1+\#\bar I}(\zeta(z),\bar I)\Big]$$

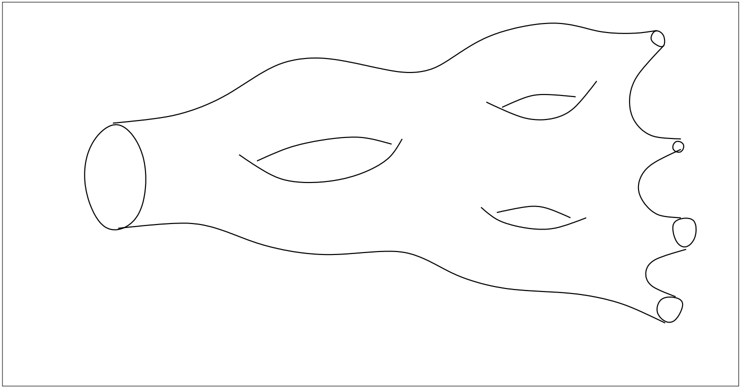
where $x'(a_i)=0$ and $x(\zeta(z))=x(z)$, and the recursion kernel is defined as

$$K(z_0,z)=\frac{\frac{1}{2}\int_{z'=\zeta(z)}^{z}\omega^{(0)}_2(z_0,z')}{\omega^{(0)}_1(z)-\omega^{(0)}_1(\zeta(z))}$$

This recursion really "computes" the generating functions. It is a recursion on the Euler characteristics $\chi_{g,n}=2-2g-n$.

# Topological recursion

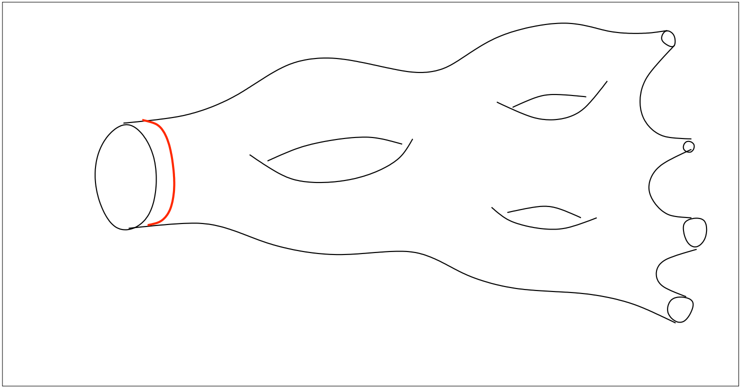
Intuitive graphical explanation:



$$\omega^{(g)}_{n+1}(z_0, \ldots, z_n)$$

# Topological recursion

Intuitive graphical explanation:



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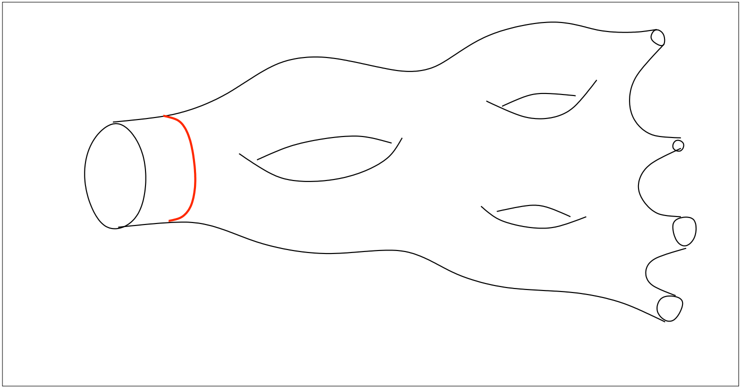

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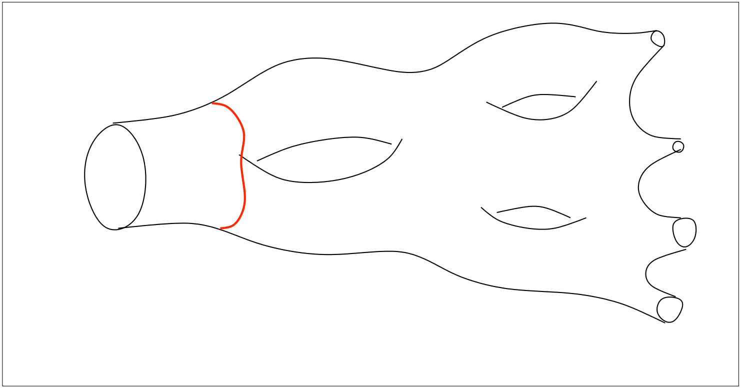

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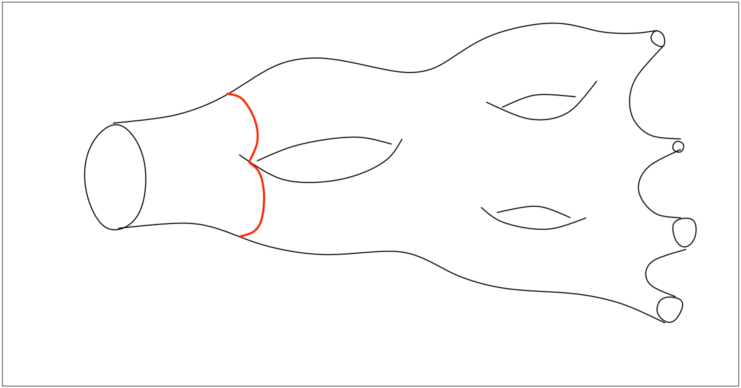

$$\omega_{n+1}^{(g)}(z_0, \ldots, z_n)$$

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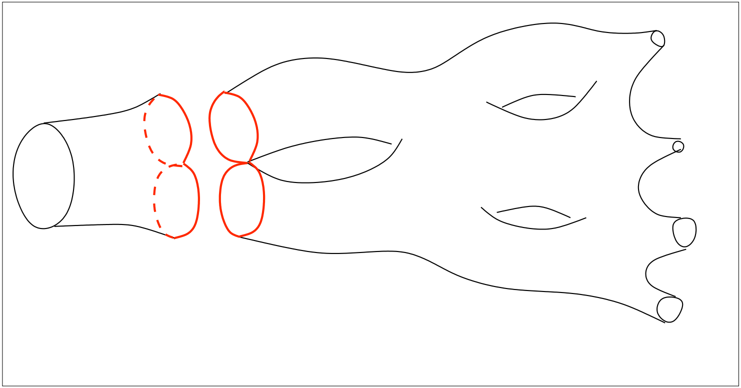
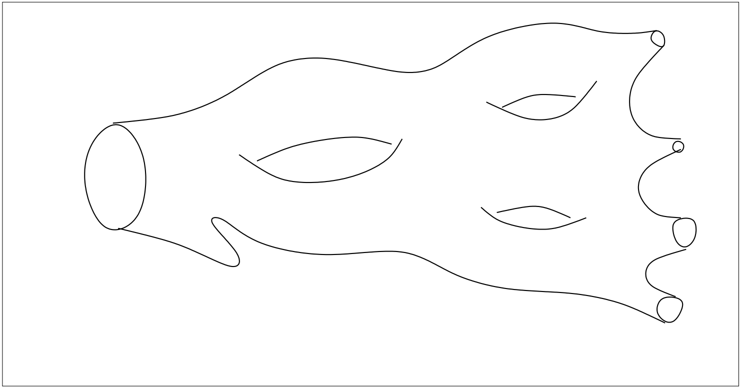


$$K(z_0, z)\ \omega^{(g-1)}_{n+2}(z, \zeta(z), z_1, \dots, z_n)$$

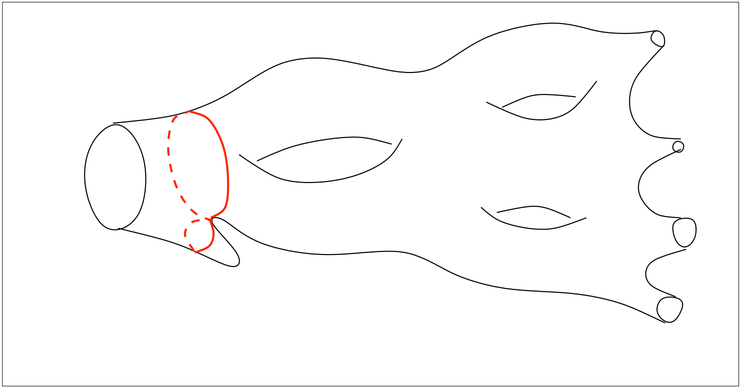
$K(z_0, z) =$ pair of pants without legs = cylinder with one side pinched.

## Intuitive graphical explanation:



$$\omega^{(g)}_{n+1}(z_0, \dots, z_n)$$

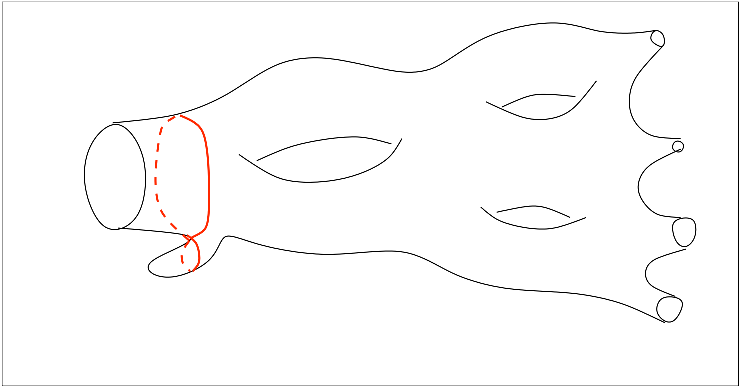
Intuitive graphical explanation:



$$K(z_0, z)\, \omega_1^{(0)}(z)\, \omega_{n+1}^{(g)}(\zeta(z), z_1, \dots, z_n)$$

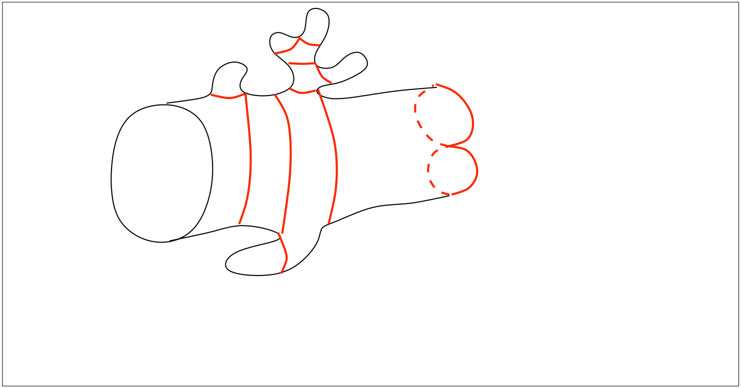
# Topological recursion

Intuitive graphical explanation:



$$K(z_0, z)\ \omega_1^{(0)}(z)\,\omega_{n+1}^{(g)}(\zeta(z), z_1, \ldots, z_n)$$

## Intuitive graphical explanation:



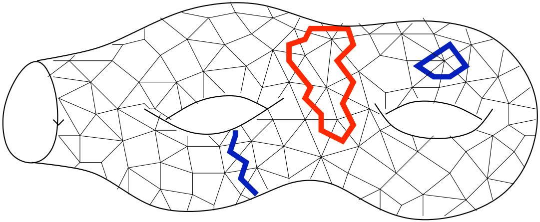
$$K(z_0, z) = \frac{\frac{1}{2}\int_{z'=\zeta(z)}^{z} \omega_2^{(0)}(z_0, z')}{\omega_1^{(0)}(z) - \omega_1^{(0)}(\zeta(z))}$$

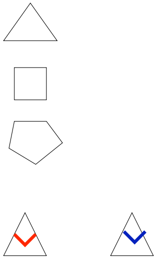
cylinder, with all possible discs

# 3. $O(\mathfrak{n})$ Model

# O(n) model

Random self avoiding loops of $\mathfrak{n}$ possible colors are drawn of the random lattice.



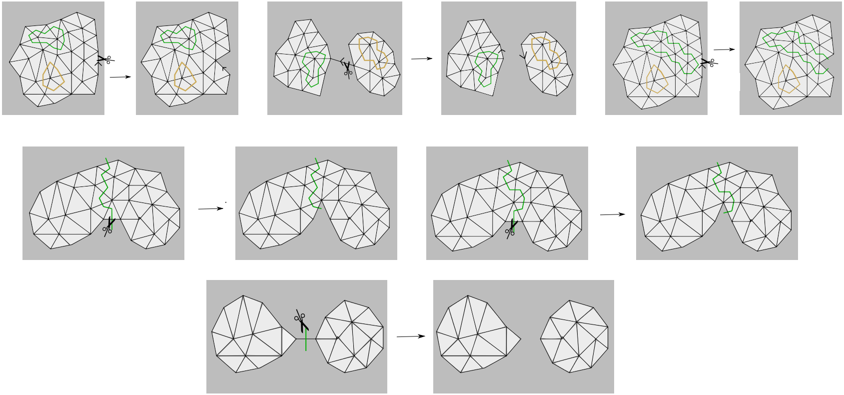


$$W_n^{(g)}(x_1,\dots,x_n;t;t_3,\dots,t_d;c;\mathfrak{n})$$

$$= \sum_v t^v \sum_{S\in\mathbb{M}_{g,n}(v)} \frac{1}{\#\mathrm{Aut}(S)} \frac{t_3^{n_3(S)}\dots t_d^{n_d(S)}}{x_1^{1+l_1(S)}\dots x_n^{1+l_n(S)}}$$

$$c^{\text{loop length}} \quad \mathfrak{n}^{\#\text{loops}}$$

"Tutte's" equations:



allow to compute all $W_n^{(g)}$'s.

# O(n) model solution

**Theorem:**
use the parametrization

$$x(z) = \frac{c}{2} + a\,\mathrm{sn}(z|\tau)$$

The 1-point function is

$$W_1^{(0)}(x) = \frac{x - \sum_j t_{2j}\, x^{2j-1}}{2-\mathfrak{n}} - \frac{\sum_j t_{2j+1}\, x^{2j}}{2+\mathfrak{n}} + A\,\frac{\prod_{j=1}^{d-1} \theta(z-\alpha_j|\tau)}{\theta(z-\frac{1}{2}-\frac{\tau}{2}|\tau)^{d-1}}$$

where the coefficients $a, \alpha_j, A$, are fixed by requiring $W_1^{(0)}(x) \sim t/x$ at large $x$, and by

$$\mathfrak{n} = -2\cos\left(2\pi \sum_j \alpha_j\right)$$

# Solution of Tutte equations, planar, 2 boundaries

We redefine the generating functions as functions of the variable $z$:

$$\omega_n^{(g)}(z_1,\dots,z_n) = W_n^{(g)}(x(z_1),\dots,x(z_n))\,x'(z_1)\dots x'(z_n) + \frac{\delta_{n,2}\delta_{g,0}\,x'(z_1)\,x'(z_2)}{(x(z_1)^2-x(z_2)^2)^2}\left(\frac{x_1^2+x_2^2}{2+\mathfrak{n}}+\frac{2x_1x_2}{2-\mathfrak{n}}\right)$$

**Theorem:** the 2-point function is universal:

$$\omega_2^{(0)}(z_1,z_2) = \wp_{\mathfrak{n}}(z_1-z_2)$$

= twisted Weierstrass function $\wp$, with a monodromy $\mathfrak{n}$.
It has a double pole at $z_1 = z_2$.

**Theorem:** [Borot-E 2009] All other "stable" topologies (i.e. $2g-2+n>0$) are given by the "Topological recursion":

$$\omega^{(g)}_{n+1}(z_0,z_1,\dots,z_n) = \sum_i \operatorname*{Res}_{z\to a_i} K(z_0,z)\Big[\omega^{(g-1)}_{n+2}(z,\zeta(z),z_1,\dots,z_n) + \sum_h \sum'_{I\uplus \bar I=\{z_1,\dots,z_n\}} \omega^{(h)}_{1+\# I}(z,I)\,\omega^{(h)}_{1+\#\bar I}(\zeta(z),\bar I)\Big]$$

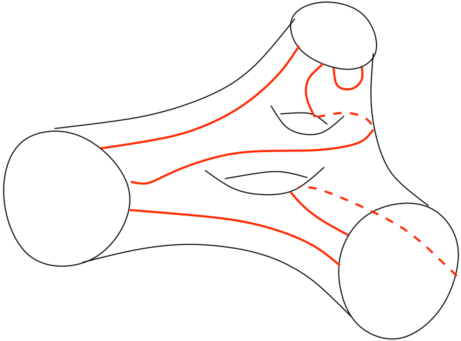
where $a_1=\frac{1}{2}$, $a_2=\frac{1+\tau}{2}$ and $x(\zeta(z))=x(z)$, and the recursion kernel is defined as

$$K(z_0,z)=\frac{\frac{1}{2}\int_{z'=\zeta(z)}^{z}\ \omega^{(0)}_2(z_0,z')}{\omega^{(0)}_1(z)-\omega^{(0)}_1(\zeta(z))}$$

Same recursion as for the Ising model !

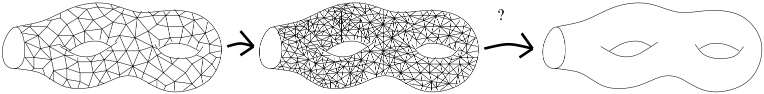
# O(n) model with boundary loops

There is a "sewing" formula (deduced from [Duplantier, Kostov 88]) to compute generating functions of O(n)model configurations with loops ending on boundaries, with some given link pattern (planar or not), given lenghts, and given lengths for the pieces of boundary.



Question: planar case, one boundary: Temperley-Lieb alebra ? (Razumov-Stroganof conjecture)

# 4. Continuum limit and Conformal Field theory




What happens when the mesh size $\epsilon \to 0$ ?

# Generalities about Continuum limits

Continuum limit:

polygons have an area $\epsilon^2$, loop pieces have length $\epsilon$. Choose $t, t_3, t_4, \dots$ such that:

- average # of polygons $\to \infty$
- area $\to$ finite $O(1)$
- lenghts $\to$ finite $O(1)$

$$\mathbb{E}(\#\text{triangles})_{g,n} = \mathbb{E}(n_3) = t_3 \frac{\partial}{\partial t_3} \ln W_n^{(g)}$$

therefore, choose $t_3, t_4, \dots$ such that $W_n^{(g)}$ = non–analytical $\to$ singularity !

Choose

$$t = t^* + \epsilon^\delta$$

such that $W_n^{(g)}$ is singular at $t = t^* = t_c$.

# Generalities about Continuum limits

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polygons have an area $\epsilon^2$, loop pieces have length $\epsilon$. Choose $t, t_3, t_4, \ldots$ such that:

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therefore, choose $t_3, t_4, \ldots$ such that $W_n^{(g)}$ = non–analytical $\to$ singularity !

Choose

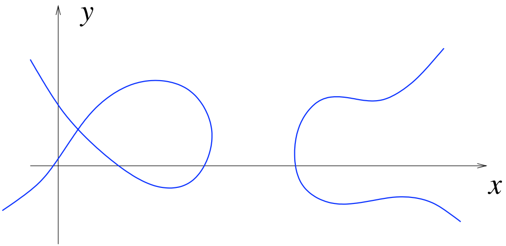
$$t_j = t_j^* + \sum_{i,j} C_{i,j} \hat{t}_j \epsilon^{\delta_j}$$

such that $W_n^{(g)}$ is singular at $t = t^* = t_c$.

# Singularities

Let $y = W_1^{(0)}(x) =$"spectral curve".

Vary $t$:



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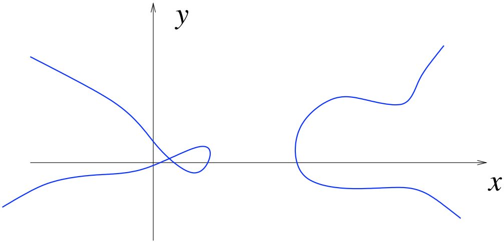

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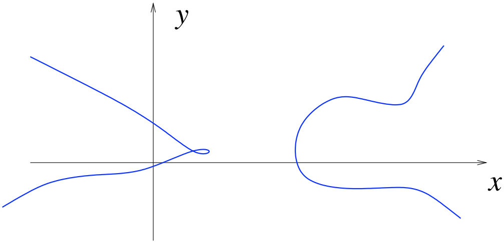

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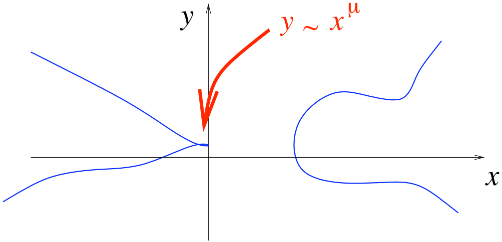


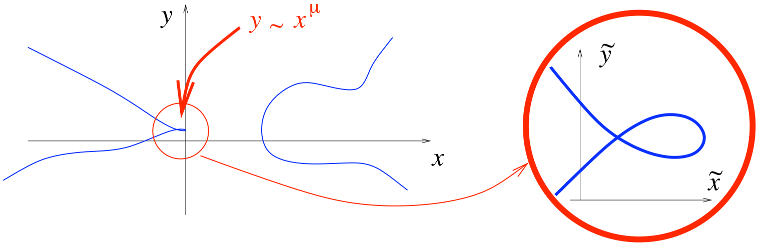

At $t = t^*$, $y$ has a cusp $y \sim x^\mu$ where

$$\mathfrak{n} = -2 \cos \mu\pi .$$

# Singularities

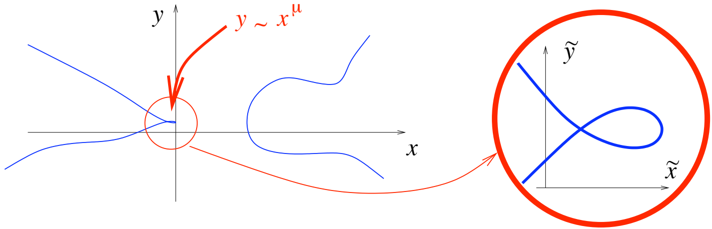
Let $y = W_1^{(0)}(x) =$"spectral curve".

Vary $t$:




At $t - t^* \sim \hat{t}\,\epsilon^2$, we rescale

$$x = x^* + \epsilon^\alpha\ \tilde{x} \qquad , \qquad y = y^* + \epsilon^{\mu\alpha}\ \tilde{y}$$




The critical spectral curve is given by:
$x = x^* + a\,\tilde{x} \quad , \quad y = y^* + a^{\mu}\ \tilde{y}$ where

$$\left\{ \begin{array}{l} \tilde{x} = -a\cosh\chi \\ \tilde{y} = \sum_{k=0}^{m} \frac{m!}{k!\,(m-k)!\,(\mu-k)}\ (2\cosh\chi)^k\ \cosh{(\mu-k)\chi} \\ a \sim (t-t^*)^{\frac{2}{\mu+1-\nu}} \sim \epsilon^{\frac{4}{\mu+1-\nu}} \end{array} \right.$$

$$\mu = 2m + 1 \pm \nu \quad , \ \nu \in [0,1[ \qquad , \ \mathfrak{n} = -2\cos\mu\pi = 2\cos\nu\pi .$$

# Scaling limits

**Theorem**

$$\exists \ \lim a^{(2g-2+n)\mu-n} W_n^{(g)}(a\tilde{x}_1,\dots,a\tilde{x}_n) = \tilde{W}_n^{(g)}(\tilde{x}_1,\dots,\tilde{x}_n)$$

and $\tilde{\omega}_n^{(g)}(\chi_1,\dots,\chi_n) = \tilde{W}_n^{(g)}(\tilde{x}_1,\dots,\tilde{x}_n)$ where $\tilde{x}_i = -\cosh\chi_i$, are given by the "topological recursion"

$$\tilde{\omega}_{n+1}^{(g)}(\chi_0,\chi_1,\dots,\chi_n) = \operatorname*{Res}_{z\to 0} \tilde{K}(\chi_0,z)\Big[\tilde{\omega}_{n+2}^{(g-1)}(z,-z,\chi_1,\dots,\chi_n) + \sum_h \sum'_{I\uplus\bar{I}=\{\chi_1,\dots,\chi_n\}} \tilde{\omega}_{1+\#I}^{(h)}(z,I)\,\tilde{\omega}_{1+\#\bar{I}}^{(h)}(-z,\bar{I})\Big]$$

where the recursion kernel is defined as

$$\tilde{K}(z_0,z) = \frac{\frac{1}{2}\int_{z'=-z}^{z} \tilde{\omega}_2^{(0)}(z_0,z')}{\tilde{\omega}_1^{(0)}(z) - \tilde{\omega}_1^{(0)}(-z)}$$

**Theorem**

$$\exists \lim a^{(2g-2+n)\mu-n} W_n^{(g)}(a\tilde{x}_1,\dots,a\tilde{x}_n) = \tilde{W}_n^{(g)}(\tilde{x}_1,\dots,\tilde{x}_n)$$

Let $\mu = p/q$, ($\mathfrak{n} = -2\cos\mu\pi$). This theorem shows that

• rescaled generating functions counting "large" maps with an $O(\mathfrak{n})$ or Ising model, tend to some "universal" functions $\tilde{\omega}_n^{(g)}$.

• The exponents $a^{(2g-2+n)\mu-n}$, together with $a \sim (t-t^*)^{\frac{2}{\mu+1-\nu}} \sim \epsilon^{\frac{4}{\mu+1-\nu}}$, are those given by the KPZ[1988] formula = conformal field theory.

• the functions $\tilde{\omega}_n^{(g)}$ satisfy some differential equations, the same as expected from Liouville CFT coupled to gravity.

# 5. General properties of the recursion

# Properties

Remark: we can apply this "topological recursion" algorithm to any plane curve $y = W_1^{(0)}(x)$ (spectral curve), (*related to a combinatorial problem or not*).

The topological recursion defines some $W_n^{(g)}$ for any plane curve, and we define:

## Definition

$F_g$ ="Symplectic Invariants" of a plane curve.

$$\forall g \geq 2, \qquad F_g = \frac{1}{2-2g} \sum_i \operatorname*{Res}_{x \to a_i} W_1^{(g)}(x)\, \Phi(x)$$

where $\Phi'(x) = W_1^{(0)}(x) = y$.

Separate definition exists for $F_0$ and $F_1$... *(but not in a 1 hour talk)*

# Properties

General properties (valid for any plane curve $y(x)$):

- $F_g =$ symplectic invariant,
- $F_g =$ (almost) modular form,
- Integrability: $Z_N = \exp\left(\sum N^{2-2g} F_g\right)(1 + \mathrm{Non.\,Pert.}) =$ Tau-function
- Limits: $F_g$ commute with limits: $\lim F_g(\mathcal{S})" = " F_g(\lim \mathcal{S})$.
This allows to study microscopic critical scaling regimes with the same method.
Ex: easily recover Tracy-Widom universal law near boundaries ($y \sim \sqrt{x}$).
Ex: recover KdV $(p,2)$ reductions near critical points of order $p$ (i.e. $y \sim x^{p/2}$), i.e. Painlevé I hierarchy.
- Many other nice properties, like special geometry deformations (form-cycle duality), Virasoro or W algebra, ...etc.

# Properties

General properties (valid for any plane curve $y(x)$):

• $F_g$ = symplectic invariant,
Theorem: if two spectral curves $(x, y)$ and $(\tilde{x}, \tilde{y})$ are such that $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$, then $F_g = \tilde{F}_g$.

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- $F_g$ = symplectic invariant,
- $F_g$ = (almost) modular form,
$F_g$+polynomial$((\mathrm{Im}\tau)^{-1})$ is modular invariant, but not analytical. Satisfies BCOV holomorphic anomaly equation.
- Integrability: $Z_N = \exp\left(\sum N^{2-2g} F_g\right)(1 + \mathrm{Non.\,Pert.}) =$ Tau-function
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  $Z_N$ satisfies formal Hirota equations. $W_n^{(g)}$ are obtained as determinants of some integrable kernel.
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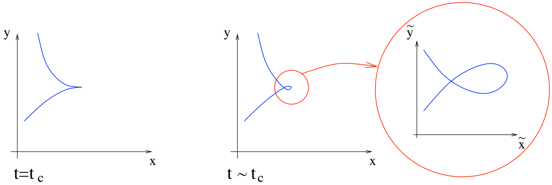
$\lim_{t\to t_c} (t-t_c)^{(2-2g)(\mu_x+\mu_y)} F_g(\mathcal{S}_t) = F_g(\tilde{\mathcal{S}}), \quad \tilde{\mathcal{S}} =$ resolved curve.

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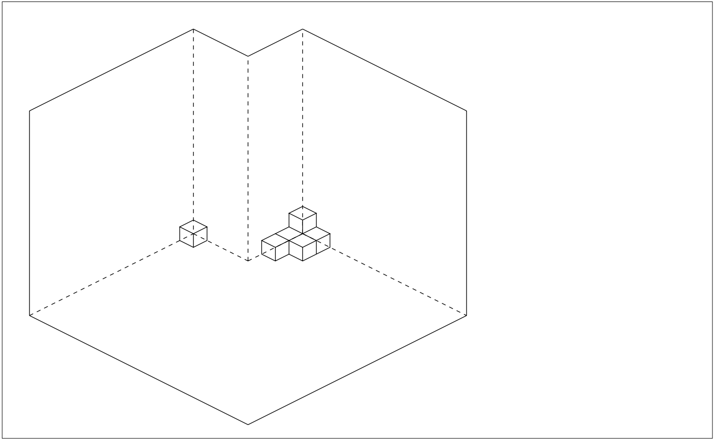
# 6. Some applications
# Beyond combinatorics of maps

# Plane partitions

- $Z = \sum_{3D \text{ partitions}} q^{\#\text{boxes}}$, $\quad q^{\text{size}} = O(1)$, large size: $q \to 1$, $\ln Z = \sum_g (\ln q)^{2g-2} \mathcal{F}_g$.



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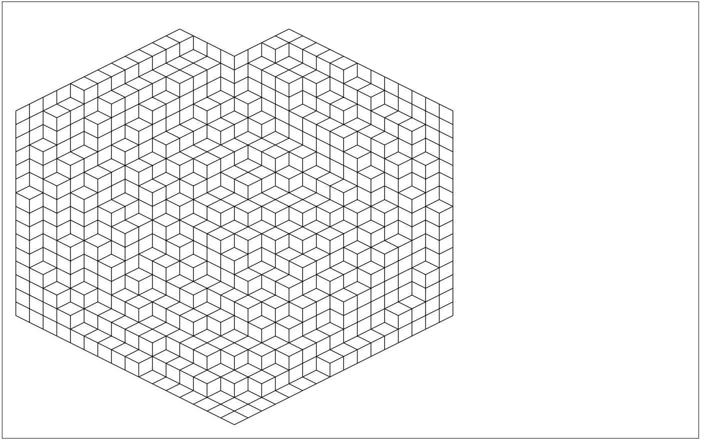

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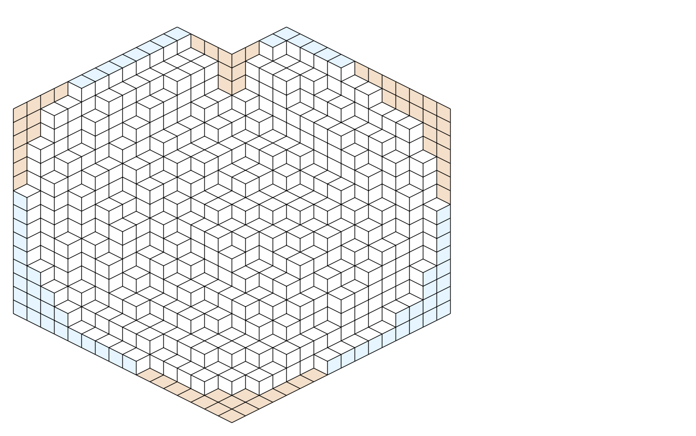

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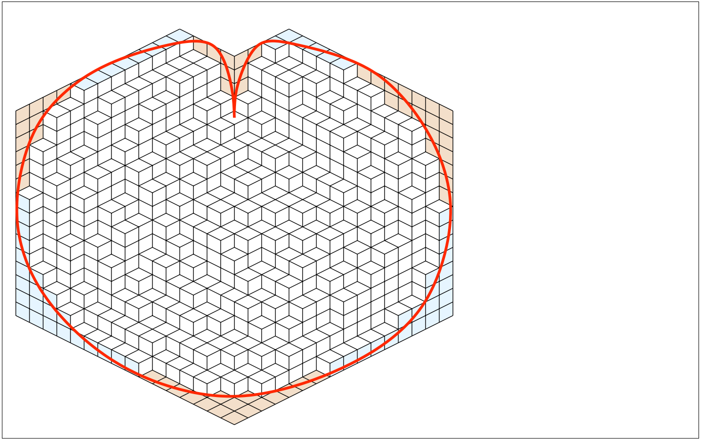

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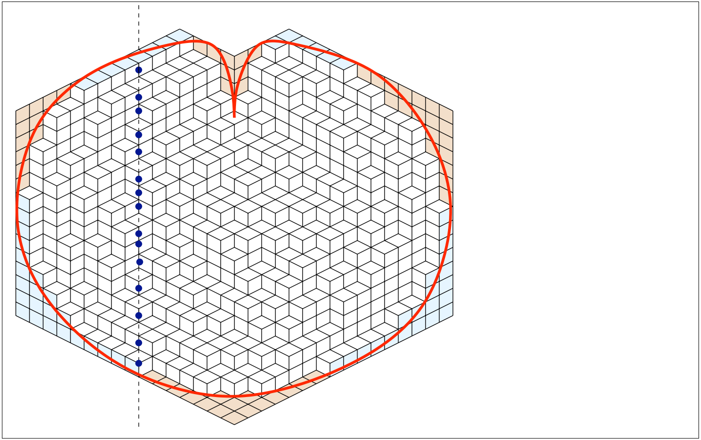

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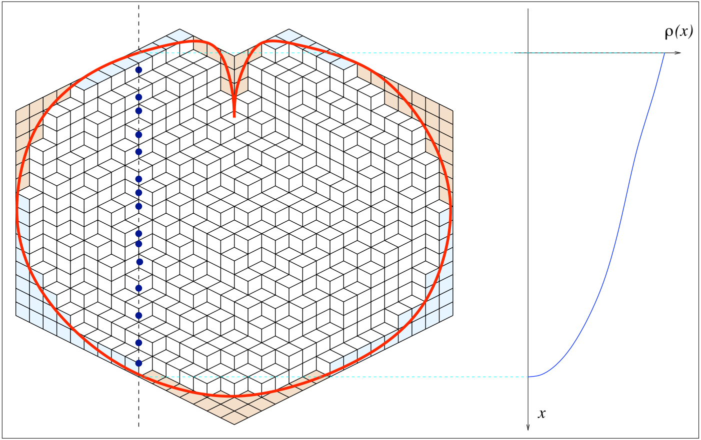


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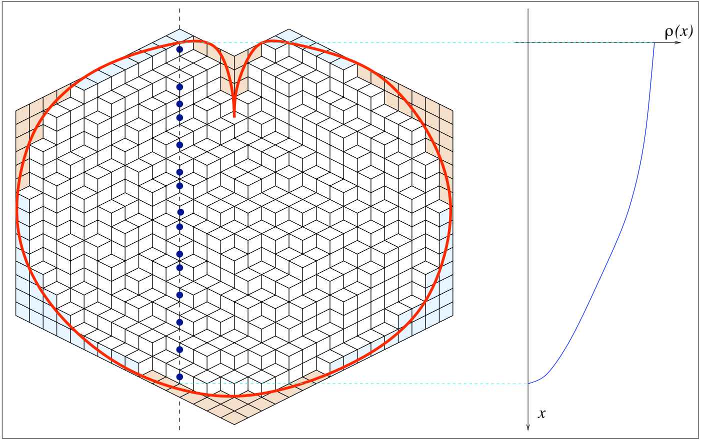


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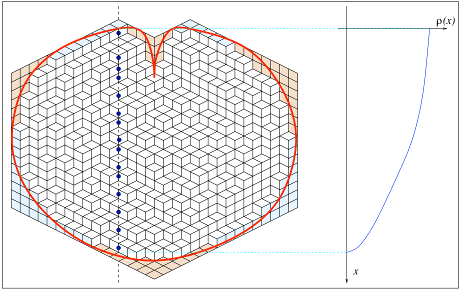


*Remark:*

*Stieljes transform of density = Legendre transform of limit shape.*

## Conjecture:

$\mathcal{F}_g = F_g$(Stieljes transf. of limit density along a vertical line) ?

*Idea of a proof:* $Z$=matrix integral, which implies that it satisfies the topological recursion. Problem: show that $W_1^{(0)} =$ Kenyon-Okounkov-Sheffield curve (limit shape) ?

# Topological strings - Gromov-Witten

• Let $\mathfrak{X}$ a 3D Calabi-Yau manifold with toric symmetry

• Gromov-Witten: $\mathcal{N}_{g,d}(\mathfrak{X}) =$ "# of conformal mappings of a Riemann surface of genus $g$ into $\mathfrak{X}$, with homology class $d$, and passing through given points".

• Generating function: $\mathcal{F}_g = \sum_d \mathcal{N}_{g,d}(\mathfrak{X})\, Q^d$.

• String theory: $\mathcal{F}_g =$ amplitude of a closed string of genus $g$ in target space $\mathfrak{X}$.

• Conjecture [Mariño 2006, BKMP 2008]:

$$\mathcal{F}_g = F_g(\text{mirror}\,\mathfrak{X})$$

Few cases proved so far:
- many low genus examples $g = 0, 1, 2, \ldots, 20$ for various choices of $\mathfrak{X}$, in particular $\mathfrak{X} =$ SW $SU(n)$ theories.
- to all genus $g = 0, \ldots, \infty$ for $\mathfrak{X} = \mathbb{C}^3$.

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# Conclusions and prospects

## Conclusion

• We have solved Tutte's equations for Ising model, $O(\mathfrak{n})$ model on a random lattice, of any topology.

• We have computed the continuum limit of generating functions $\rightarrow$ compatible with CFT. (exponents = KPZ).

• Extension to other combinatorial or algebraic problems (Gromov-Witten theory, plane partitions, random matrices...).

## Some open questions

• can we compute generating functions of configurations with points at fixed distance (metrics properties) ? Idea: fix points as marked faces of zero size, then count configurations with loops of given lengths, between those marked faces...

• Prove that the topological recursion computes plane partitions, Gromov-Witten invariants...

# Some references

Book in preparation:
draft can be found at
http://eynard.bertrand.voila.net/TOCbook.htm

• For the O(n) model:
G. Borot, B. Eynard, Enumeration of maps with self avoiding loops and the O(n) model on random lattices of all topologies, math-ph: arxiv.0910.5896. J. Stat. Mech. (2011) P01010.

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B. E., N. Orantin, Mixed correlation functions in the 2-matrix model, and the Bethe ansatz, JHEP/0508 (2005) 028, hep-th/0504029.