Benjamin-Ono integrable systems in 2D conformal field theory

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Plan of talk:

1. Integrals of Motion in 2D CFT

2. Classical BO equation: review

3. Quantization of BO system: relation to CS model

4. Applications: conformal blocks and AGT formula

5. Concluding remarks

Integrals of Motion in 2D CFT:

- Local Integrals of Motion were introduced by Zamolodchikov in 1987

- Consider CFT with the symmetry algebra $\mathcal{A}$ and define

$$\mathbf{I}_k = \frac{1}{2\pi} \int \mathbf{G}_{k+1} dx$$

  such that

  1. $[\mathbf{I}_k, \mathbf{I}_l] = 0$

  2. $\mathbf{I}_k$ has spin k

  3. The simultaneous spectrum of $\mathbf{I}_k$ is non-degenerate

  4. Some other conditions: like appropriate semiclassical limit

- The most known example is the system originally studied by Zamolodchikov: $\mathcal{A} = \text{Vir}$, $T(z)$ with the central charge c

$$\mathbf{G}_2 = T(z), \quad \mathbf{G}_4 = (T(z))^2, \quad \mathbf{G}_6 = (T(z))^3 + \frac{c+2}{12}(T'(z))^2, \ \ldots$$

- In semiclassical limit $c \to \infty$ it reduces to KdV system

- One of the advantages of IM's is that they may survive under the integrable perturbation ($\Phi_{1,3}$ perturbation in this particular case)

- Bazhanov, Lukyanov and Zamolodchikov studied this system in great details. In particular they derived T and Q functions ...

- One of the impressive results is the so called IM/ODE correspondence (Dorey, Tateo, BLZ)

Benjamin-Ono equation:

- Appears in hydrodynamics of stratified fluid (Benjamin–1967, Ono–1975)

$$\mathsf{v}_t + 2\mathsf{v}\mathsf{v}_x + \mathsf{H}\mathsf{v}_{xx} = 0, \qquad x \in [0, 2\pi]$$

where H is the operator of Hilbert transform defined by

$$\mathsf{H}F(x) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} F(y)\cot\frac{1}{2}(y-x)\,dy.$$

- Following Bock-Kruskal and Nakamura (1979) we define

$$\mathsf{v} = \lambda\,(e^w - 1) + i\,\mathsf{P}_+\,w_x, \qquad \mathsf{P}_\pm = \frac{1}{2}(1 \mp i\,\mathsf{H}) \tag{$\star$}$$

Substituting ($\star$) into BO equation one arrives at

$$\left(\lambda e^w + i\,\mathsf{P}_+\frac{\partial}{\partial x}\right)\left[w_t + 2\mathsf{v}w_x - iw_x^2 + \mathsf{H}w_{xx}\right] = 0$$

- We can assume that the expression in angular brackets vanishes

$$w_t + 2\mathsf{v}w_x - iw_x^2 + \mathsf{H}w_{xx} = 0$$

- Substituting $w = w^+ + w^-$, where $w^\pm = \mathsf{P}_\pm w$ one finds

$$\begin{aligned} &i\phi_x^+ + (\lambda + \mathsf{v})\phi^+ = \lambda\phi^-, \\ &\phi_t^\pm - i\phi_{xx}^\pm - 2\lambda\phi_x^\pm \mp 2\phi^\pm \mathsf{P}_\pm \mathsf{v}_x = 0 \end{aligned} \tag{*}$$

  where

$$\phi^+ = e^{-w^+}, \quad \phi^- = e^{w^-}.$$

  We note that this system is a Lax pair for BO equation.

- One can easily check that w is a conserved density

$$\frac{\partial}{\partial t}\int_0^{2\pi} w(x,t)\,dx = 0,$$

  and hence its expansion in spectral parameter λ gives infinitely conserved densities for BO equation.

- Expanding at $\lambda \to \infty$ (here and below $\mathsf{D} = \mathsf{H}\frac{d}{dx}$) we get

$$G_1 = \mathsf{v}, \;\; G_2 = \frac{\mathsf{v}^2}{2}, \;\; G_3 = \frac{\mathsf{v}^3}{3} + \frac{1}{2}\mathsf{v}\mathsf{D}\mathsf{v}, \;\; G_4 = \frac{\mathsf{v}^4}{4} + \frac{3}{4}\mathsf{v}^2\mathsf{D}\mathsf{v} + \frac{1}{2}\mathsf{v}_x^2, \ldots$$

such that the quantities (classical Integrals of Motion)

$$I_k \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} G_{k+1}\,dx,$$

are conserved in time.

- We note that BO equation can be written in a Hamiltonian form

$$\mathsf{v}_t = \{I_2, \mathsf{v}\}, \qquad \{\mathsf{v}(x), \mathsf{v}(y)\} = \delta'(x-y),$$

and all the quantities I_k form a commutative Poisson algebra

$$\{I_k, I_l\} = 0.$$

- All this can be generalized for the $n-$matrix version of BO equation. For example for $n = 2$ we have Lax pair

$$\Big((\partial - iv + \lambda)^2 + u\Big)\phi^+ = \lambda^2\phi^-,$$
$$\phi_t^\pm - i\lambda\phi_x^\pm - \frac{i}{2}\phi_{xx}^\pm \mp 2\phi^\pm \, \mathsf{P}_\pm v_x = 0,$$

- The compatibility condition is equivalent to BO_2 equation

$$\begin{cases} u_t + vu_x + 2uv_x + \frac{1}{2}v_{xxx} = 0, \\ v_t + \frac{u_x}{2} + \mathsf{H}v_{xx} + vv_x = 0, \end{cases}$$

- This equation is also Hamiltonian with $\mathcal{H} = 1/2\pi \int (uv + v\mathsf{D}v + 1/3\, v^3)dx$

$$\{u(x), u(y)\} = (u(x) + u(y))\,\delta'(x-y) + \frac{1}{2}\delta'''(x-y),$$
$$\{v(x), v(y)\} = \frac{1}{2}\delta'(x-y), \quad \{u(x), v(y)\} = 0.$$

Quantization of BO system: relation to CS model

- the field $\mathsf{v}(x)$ is the semiclassical limit of the $U(1)$ current field

$$v(z) = P + \sum_{k \neq 0} a_k e^{-ikz},$$

defined on a half-cylinder.

- The Fourier components a_n satisfy

$$[a_m, a_n] = m\, \delta_{m,-n}.$$

- The quantum counter part of the integral I_2 has to be chosen

$$\mathbf{I}_2 = \frac{1}{2\pi} \int_0^{2\pi} : \left(\frac{v^3}{3} + \frac{iQ}{2} v \mathsf{D} v \right) : dx,$$

where $Q = b + 1/b$ and the semiclassical limit $b \to 0$ is realized as

$$v \to -ib^{-1} \mathsf{v}.$$

- In components we have

$$\mathbf{I}_2 = iQ\sum_{k>0} k a_{-k} a_k + \frac{1}{3}\sum_{i+j+k=0} a_i a_j a_k$$

- One can represent

$$a_{-k} = -ib\, p_k, \qquad a_k = \frac{i}{b}\, k \frac{\partial}{\partial p_k},$$

where p_k are power-sum symmetric polynomials

$$p_k = p_k(x) = \sum_{j=1}^{N} x_j^k.$$

- When acting on a space of symmetric functions the operator $\mathbf{I}_2$ can be rewritten as (Awata et al 1995)

$$\mathbf{I}_2 \sim \sum_{i=1}^{N}\left(x_i \frac{\partial}{\partial x_i}\right)^2 + g\sum_{i<j}\frac{x_i + x_j}{x_i - x_j}\left(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j}\right) = \mathcal{H}_2^{\mathsf{CS}},$$

- It is natural to assume that higher IM's $\mathbf{I}_k$ can be expressed in terms of higher Calogero-Sutherland Hamiltonians:

$$\mathbf{I}_k \sim \mathcal{H}_k^{\mathsf{CS}}$$

- This statement can be checked by explicit computation on lower levels, but the general proof as well as the explicit form of the relation between two integrable systems is still lacking

- In fact higher qBO systems are also related to CS model

Applications: conformal blocks and AGT formula:

- We consider CFT with symmetry algebra being $\mathcal{A} = W_n \otimes H$

$$v(z),\ W^{(2)}(z) = T(z),\ W^{(3)}(z), \dots W^{(n)}(z)$$

- In universal enveloping of $\mathcal{A}$ we can construct system of IM's $\mathbf{I}_k$:

$$\mathbf{I}_k = \frac{1}{2\pi} \int \mathbf{G}_{k+1} \left[v, W^{(k)}, \mathsf{D} \right] dx, \qquad [\mathbf{I}_k, \mathbf{I}_l] = 0$$

such that in semiclassical limit $b \to \infty$, $v \to b^{-1}\mathsf{v}$, $W^{(k)} \to b^{-k}\mathsf{u}_k$

$$\mathbf{I}_k \to I_k^{\mathsf{BO}_n}$$

- For example for $n = 2$ we have $\mathcal{A} = \mathsf{Vir} \otimes H$ with $c = 1 + 6Q^2$

$$\mathbf{I}_3 = \frac{1}{2\pi} \int \left(Tv - iQ\, v\mathsf{D}v + \frac{1}{3} v^3 \right) dx$$

- We can find simultaneous eigenfunctions

$$\mathbf{I}_k|P\rangle_{\vec{\lambda}} = h^{(k)}_{\vec{\lambda}}(P)|P\rangle_{\vec{\lambda}},$$

where $P = (P_1, P_2, \dots, P_{n-1})$ and $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ - n partitions

$$|P\rangle_{\vec{\lambda}} = \Big((-1)^{(n-1)|\vec{\lambda}|}(W^{(n)}_{-1})^{|\vec{\lambda}|} + \dots\Big)|P\rangle$$

- Then we define primary operator

$$V_a = e^{(nQ-a)/\sqrt{n}\phi_-}e^{a/\sqrt{n}\phi_+}V^W_{a\omega_1},$$

where $\phi_\pm = \sum_{k\in\pm} ia_k/ke^{-ikz}$ and $V^W_{a\omega_1}$ is the primary field in W_n theory

- and matrix elements

$$\mathcal{F}(a, P, \lambda; P', \nu) =_{\vec{\nu}} \langle P'|V_a|P\rangle_{\vec{\lambda}}$$

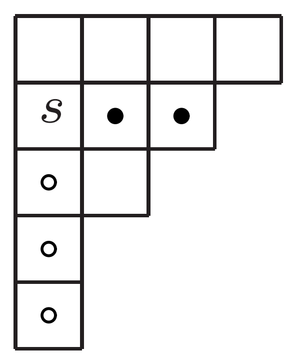
- One can prove (Alba et al 2010, Fateev and A.L. 2011) that

$$\mathcal{F}(a,P,\lambda;P',\nu) =$$
$$= \prod_{i,j=1}^{n} \prod_{s\in\lambda_i} (Q - E_{\lambda_i,\nu_j}(x_j - x_i'|s) - a/n) \prod_{t\in\nu_j} (E_{\nu_j,\lambda_i}(x_i' - x_j|t) - a/n),$$

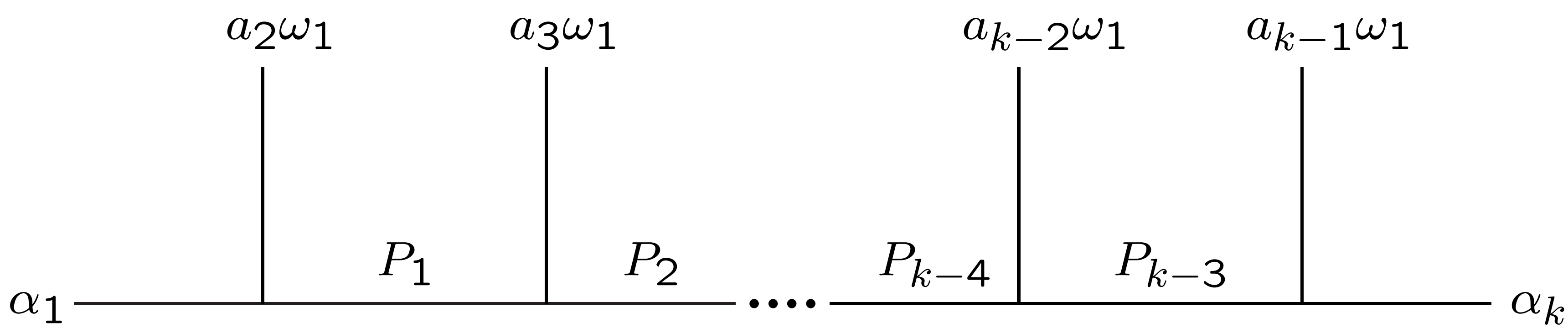
where $x_j = (P, h_j)$ with $h_j = \omega_1 - e_1 - \cdots - e_{j-1}$ and

$$E_{\lambda,\mu}\Big(P\Big|s\Big) = P - b\, l_\mu(s) + b^{-1}(a_\lambda(s) + 1).$$

where $a_\lambda(s)$ and $l_\mu(s)$ are the arm length of the square s in the partition λ and the leg length of the square s in the partition μ



Consider W_n CFT with $c = (n-1)(1+n(n+1)Q^2)$. The primary fields V_α are parameterize by vector parameters $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$. Then we consider the conformal block of special form:



It is also convenient to choose $z_1 = 0$, $z_{k-1} = 1$, $z_k = \infty$ and

$$z_{i+1} = q_i q_{i+1} \ldots q_{k-3} \quad \text{for} \quad 1 \le i \le k-3.$$

The AGT relation claims that the function

$$\mathbb{Z}(q) \overset{\text{def}}{=} \prod_{j=1}^{k-3} \prod_{m=j}^{k-3} (1 - q_j \ldots q_m)^{a_{j+1}(Q - a_{m+2}/n)} \, \mathbb{F}(q) = 1 + \sum_{\vec{j}} q_1^{j_1} q_2^{j_2} \ldots q_{k-3}^{j_{k-3}} \, \mathbb{Z}_{\vec{j}},$$

coincides the instanton part of the Nekrasov partition function.

Concluding remarks:

- Multiplying W_n theory by additional free boson we found an integrable system with nice properties. In particular, it helps with computation of the conformal blocks.

- Can we find similar phenomena for different CFT's? Yes, as was predicted by (Shiraishi, Feigin et al 2011)

- We find the generalization for NS algebra. Again BO system. (Belavin, Beshtein, Tarnopolsky, A.L. to appear)

- It seems that we can do the same for parafermionic “NS” algebra...

- The physical=conceptual understanding is far from being complete