Random Convex Hull and Extreme Value Statistics

Satya N. Majumdar

Laboratoire de Physique Théorique et Modèles Statistiques,CNRS,
Université Paris-Sud, France

September 15, 2011

Collaborators:

A. Comtet (LPTMS, Orsay, FRANCE)
J. Randon-Furling (Univ. Paris-1, FRANCE)

Ref: Phys. Rev. Lett. 103, 140602 (2009)

Extended Review: J. Stat. Phys. 138, 955 (2010)

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Recent work with A. Reymbaut and A. Rosso (LPTMS, Orsay, France)
arXiv:1108.5455 (to appear in J. Phys. A: Math. Theor. (2011))

Plan

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- Random Convex Hull $\Longrightarrow$ definition

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- Random Convex Hull $\Longrightarrow$ definition
- Convex Hull of n planar Brownian motions

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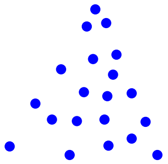
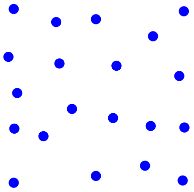
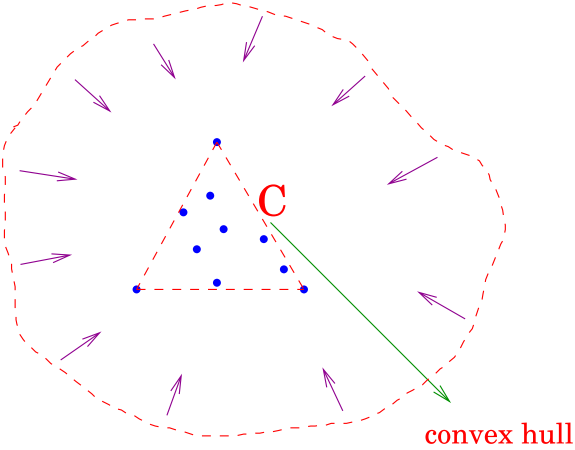
- Exact results for the mean perimeter and the mean area for all n.
- Summary and Conclusions

Shape of a set of Points

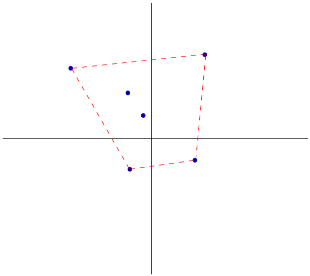


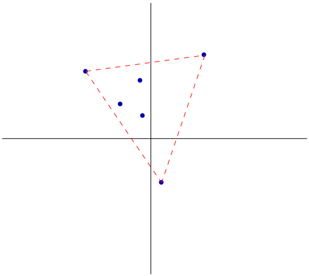


Shape of a set of Points: Convex Hull



Random Convex Hull in a Plane



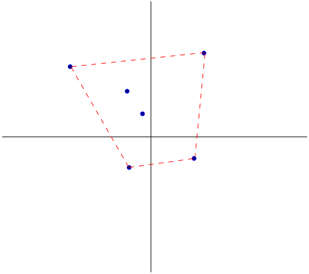


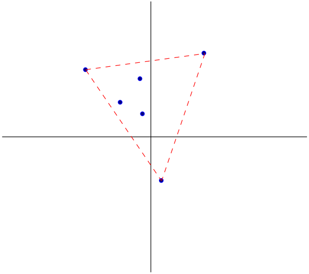
- Convex Hull $\Longrightarrow$ Minimal convex polygon enclosing the set

Random Convex Hull in a Plane

- Convex Hull $\Longrightarrow$ Minimal convex polygon enclosing the set
- The shape of the convex hull $\rightarrow$ different for each sample

Random Convex Hull in a Plane





- Convex Hull $\Longrightarrow$ Minimal convex polygon enclosing the set
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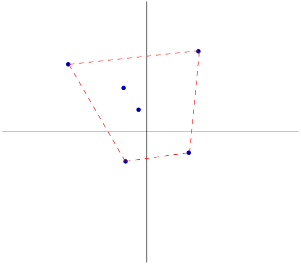
- Points drawn from a distribution $P(\vec{r_1}, \vec{r_2}, \ldots, \vec{r_N})$
  $\rightarrow$ Independent or Correlated

Random Convex Hull in a Plane

- Convex Hull $\Longrightarrow$ Minimal convex polygon enclosing the set
- The shape of the convex hull $\rightarrow$ different for each sample
- Points drawn from a distribution $P(\vec{r_1}, \vec{r_2}, \ldots, \vec{r_N})$
  $\rightarrow$ Independent or Correlated
- Question: Statistics of observables: perimeter, area and no. of vertices

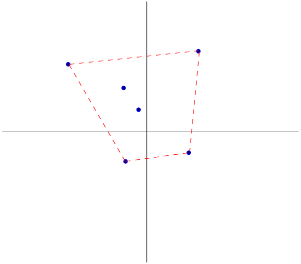
Independent Points in a Plane



Each point chosen independently from the same distribution

$$P\left(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N\right) = \prod_{i=1}^{N} p(\vec{r}_i)$$

Independent Points in a Plane



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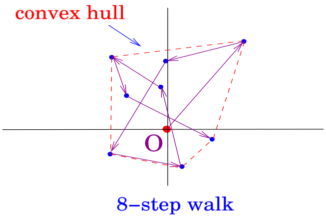
Associated Random Convex Hull $\rightarrow$ well studied by diverse methods

Lévy ('48), Geffroy ('59), Spitzer & Widom ('59), Baxter ('59)
Rényi & Sulanke ('63), Efron ('65), Molchanov ('07)....many others

Correlated Points: Vertices of an Open Random Walk



Discrete-time random Walk of N steps

$x_k = x_{k-1} + \xi_x(k)$
$y_k = y_{k-1} + \xi_y(k)$

$\xi_x(k), \xi_y(k) \rightarrow$ Independent jump lengths

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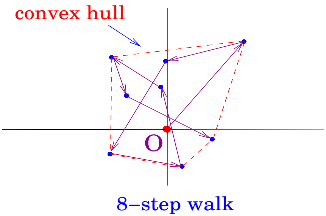

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Continuous-time limit: Brownian path of duration T

$\frac{dx}{d\tau} = \eta_x(\tau)$

$\frac{dy}{d\tau} = \eta_y(\tau)$

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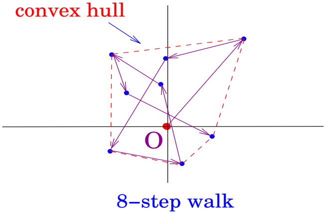

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$$\langle \eta_x(\tau)\eta_x(\tau')\rangle = 2\, D\, \delta(\tau - \tau')$$

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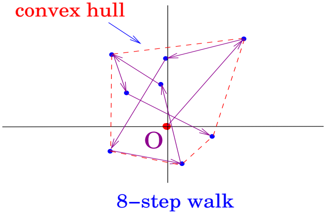

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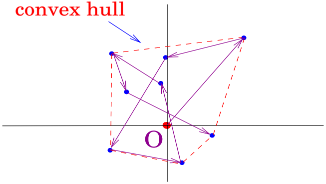
Correlated Points: Vertices of an Open Random Walk



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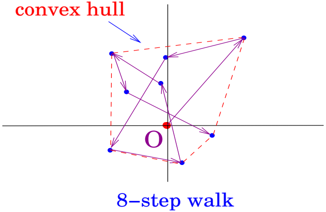
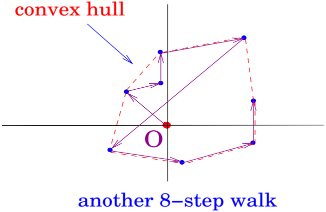
$$\langle \eta_x(\tau)\eta_y(\tau') \rangle = 0$$

Correlated Points: Vertices of an Open Random Walk





- Continuous-time limit: Brownian path of duration T

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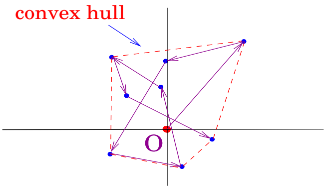
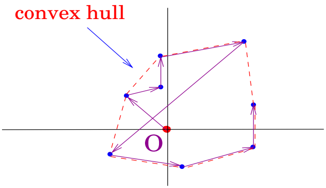


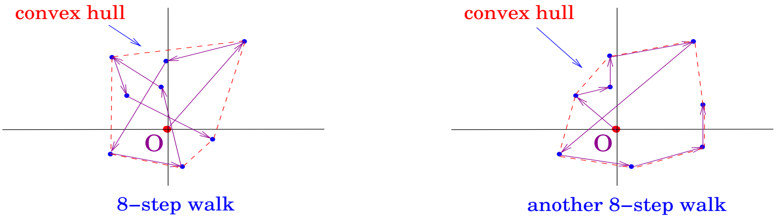
8–step walk



another 8–step walk

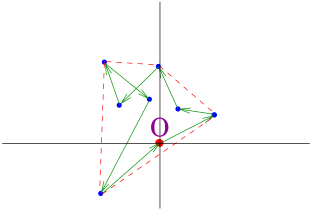
- Continuous-time limit: Brownian path of duration T
- mean perimeter and mean area of the associated Convex hull?

Correlated Points: Vertices of an Open Random Walk

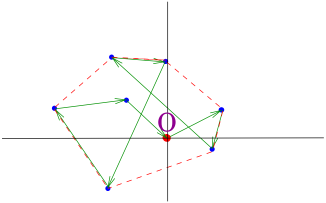


- Continuous-time limit: Brownian path of duration T
- mean perimeter and mean area of the associated Convex hull?
- mean perimeter: $\langle L_1 \rangle = \sqrt{8\pi}\,\sqrt{2\,D\,T}$ (Takács, '80)
- mean area: $\langle A_1 \rangle = \frac{\pi}{2}\,(2\,D\,T)$ (El Bachir, '83, Letac '93)

Correlated Points: Vertices of a Closed Random Walk



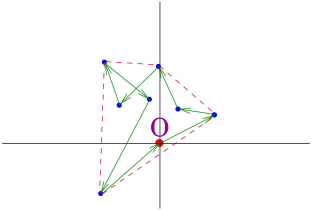
8 step random bridge



another 8 step bridge

• Continuous-time limit: Brownian bridge of duration T : starting at O and returning to it after time T

Correlated Points: Vertices of a Closed Random Walk



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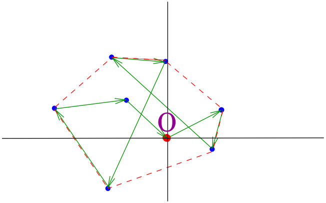

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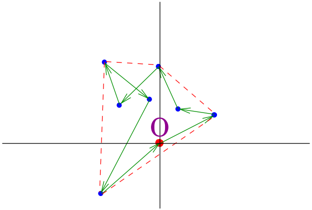

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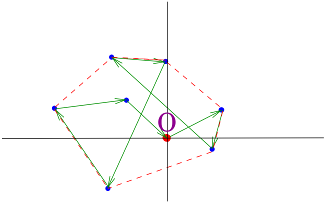

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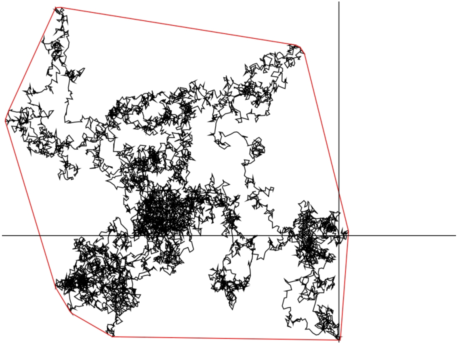
- mean area: $\langle A_1 \rangle = (?)\,(2\,D\,T)$

Home Range Estimate via Convex Hull



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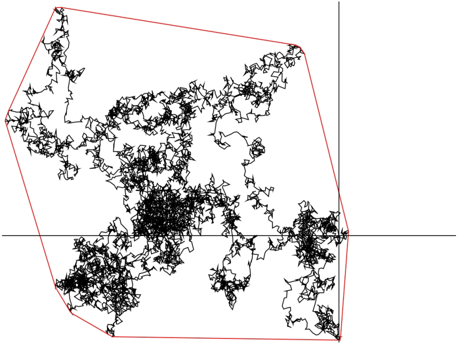

Models of home range for animal movement, Worton (1987)
Integrating Scientific Methods with Habitat Conservation Planning, Murphy and Noon (1992)
Theory of home range estimation from displacement measurements of animal populations, Giuggioli et. al. (2005)
Home Range Estimates, Boyle et. al., (2009)

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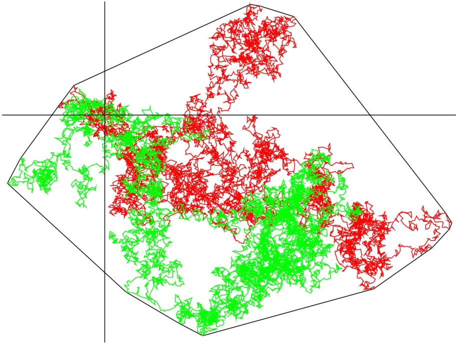

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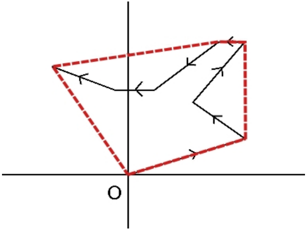
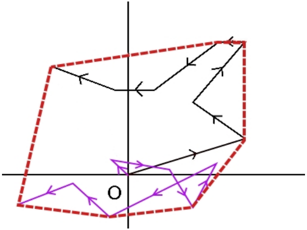
Global Convex Hull of n Independent Brownian Paths





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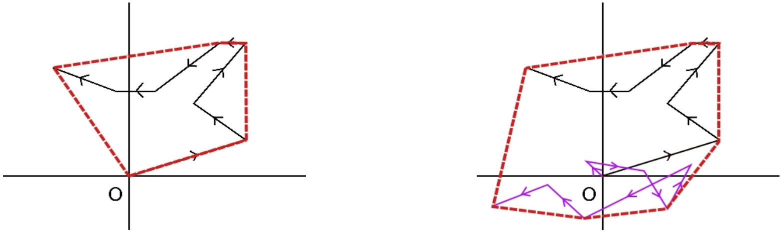
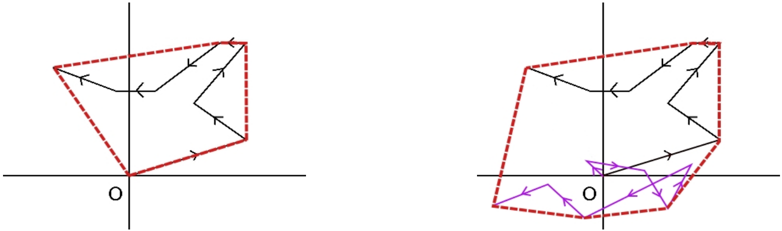


• Mean perimeter $\langle L_n \rangle$ and mean area $\langle A_n \rangle$ of n independent Brownian paths (bridges) each of duration T?

Global Convex Hull of n Independent Brownian Paths



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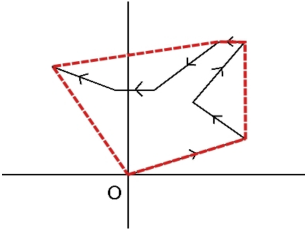
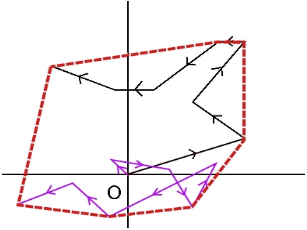
- $\langle L_n \rangle = \alpha_n \sqrt{2\,D\,T}$; $\qquad \langle A_n \rangle = \beta_n \,(2\,D\,T)$

Global Convex Hull of n Independent Brownian Paths





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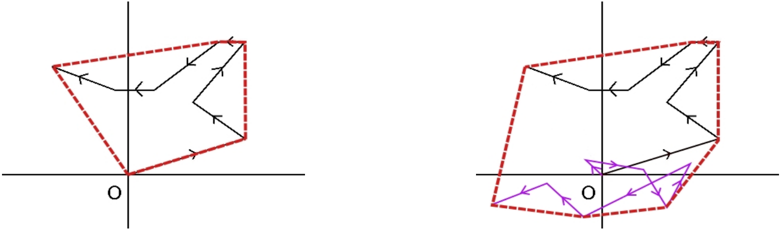
- Recall $\boxed{\alpha_1 = \sqrt{8\pi}, \;\; \beta_1 = \pi/2}$ (open path)

  $\boxed{\alpha_1 = \sqrt{\pi^3/2}, \;\; \beta_1 = ?}$ (closed path)

Global Convex Hull of n Independent Brownian Paths



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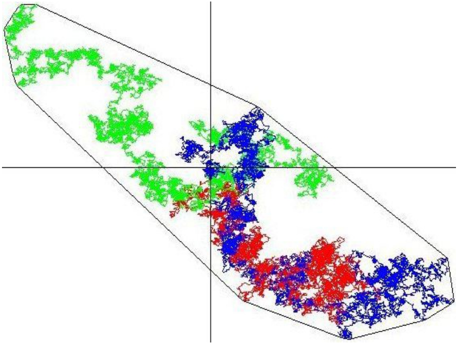
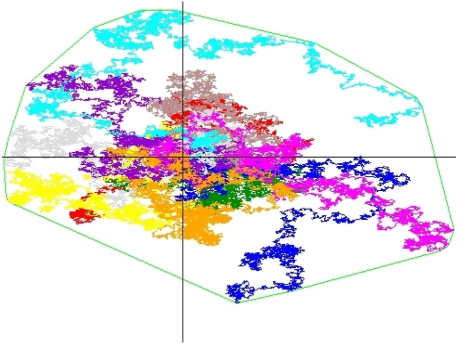
  $\boxed{\alpha_1 = \sqrt{\pi^3/2}, \quad \beta_1 = ?}$ (closed path)
- $\boxed{\alpha_n, \quad \beta_n = ?}$ $\rightarrow$ both for open and closed paths $\rightarrow$ n-dependence?

Global Convex Hull of n Independent Brownian Paths

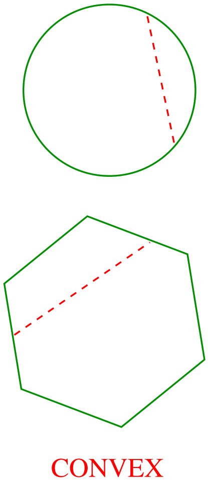


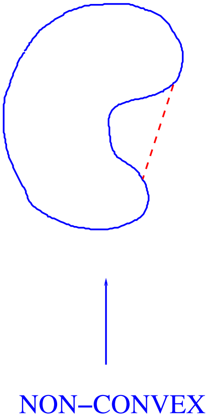
$n = 3$ closed paths



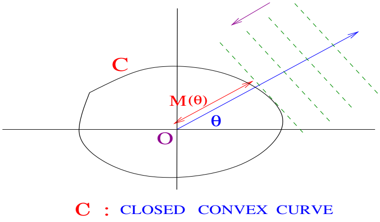
$n = 10$ open paths

Closed Convex Curves





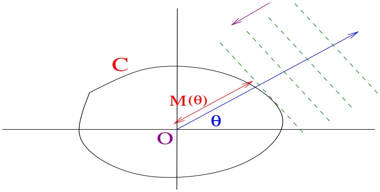
Cauchy's Formulae for a Closed Convex Curve



- For any point $[X(s), Y(s)]$ on C define:

  Support function: $M(\theta) = \max_{s \in C} [X(s)\cos(\theta) + Y(s)\sin(\theta)]$

Cauchy's Formulae for a Closed Convex Curve



C : CLOSED CONVEX CURVE

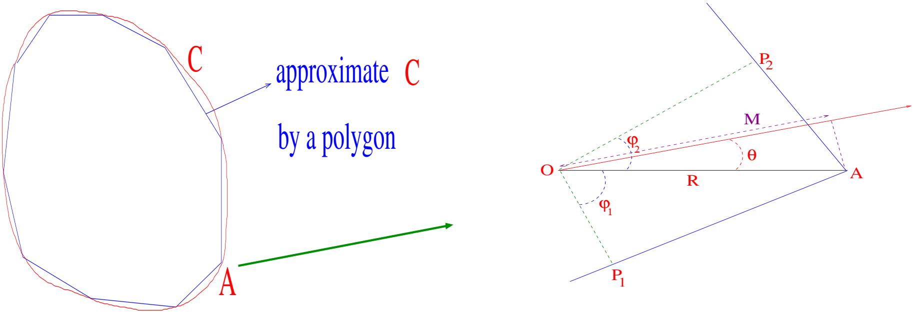
- For any point $[X(s), Y(s)]$ on C define:

Support function: $M(\theta) = \max_{s \in C} [X(s)\cos(\theta) + Y(s)\sin(\theta)]$

- Perimeter: $$L = \int_0^{2\pi} d\theta \; M(\theta)$$ (A. Cauchy, 1832)

- Area: $$A = \frac{1}{2}\int_0^{2\pi} d\theta \; \left[M^2(\theta) - [M'(\theta)]^2\right]$$

A simple physicist's proof of Cauchy's formula



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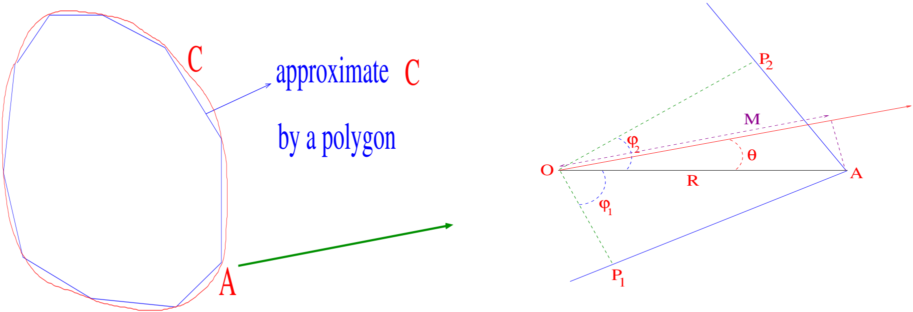
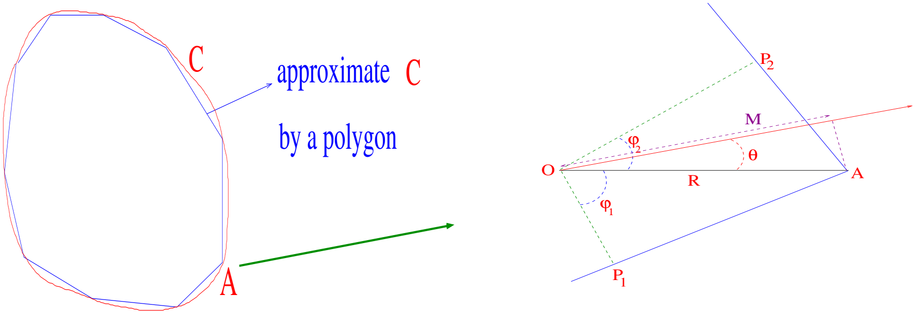


$$M(\theta) = R\cos\theta$$

A simple physicist's proof of Cauchy's formula



$$\boxed{M(\theta) = R\cos\theta}$$

Perimeter: $\int_{-\phi_1}^{\phi_2} M(\theta) d\theta = R\left[\sin(\phi_1) + \sin(\phi_2)\right] = L_{P_1 A P_2}$

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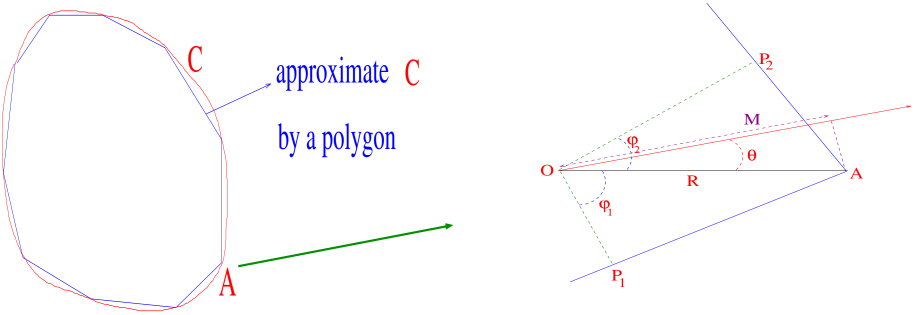

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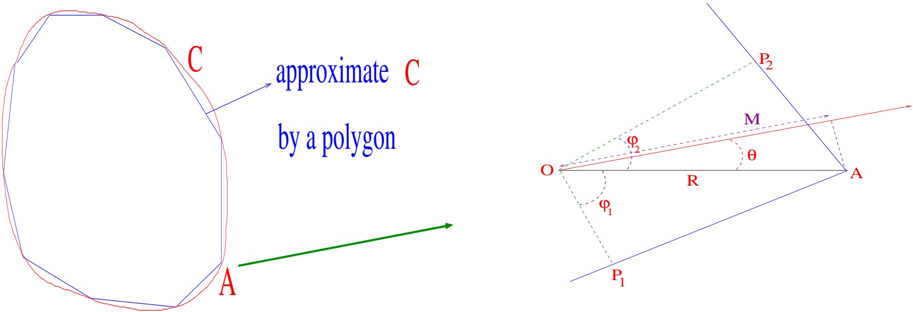
A simple physicist's proof of Cauchy's formula



$$M(\theta) = R\cos\theta$$

Perimeter: $\int_{-\phi_1}^{\phi_2} M(\theta)d\theta = R\left[\sin(\phi_1) + \sin(\phi_2)\right] = L_{P_1AP_2}$

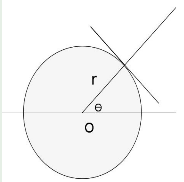
Area: $\frac{1}{2}\int_{-\phi_1}^{\phi_2}\left[M^2(\theta) - (M'(\theta))^2\right] d\theta$

$$= \frac{R^2}{2}\left[\sin(\phi_2)\cos(\phi_2) + \sin(\phi_1)\cos(\phi_1)\right] = A_{OP_1AP_2}$$

Examples

a circle centered at the origin:

$$M(\theta) = r$$



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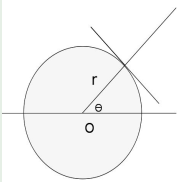

$$\begin{aligned} L &= \int_0^{2\pi} d\theta \; M(\theta) = 2\pi r \\ A &= \frac{1}{2}\int_0^{2\pi} d\theta \; \left[M^2(\theta) - [M'(\theta)]^2 \right] = \pi r^2 \end{aligned}$$

Examples

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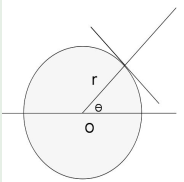
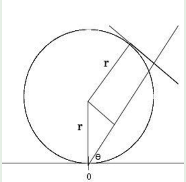


a circle touching the origin:

$$M(\theta) = r(1 + \sin\theta)$$



Examples

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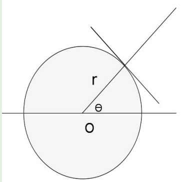

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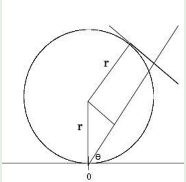

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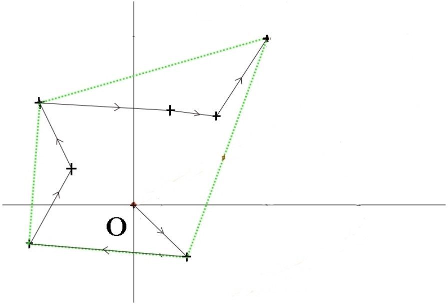
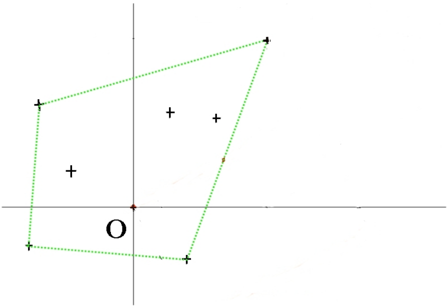
Cauchy's formulae Applied to Convex Polygon



Let $(x_k, y_k) \Longrightarrow$ vertices of an N-step random walk starting at O

Let C (green) be the associated Convex Hull

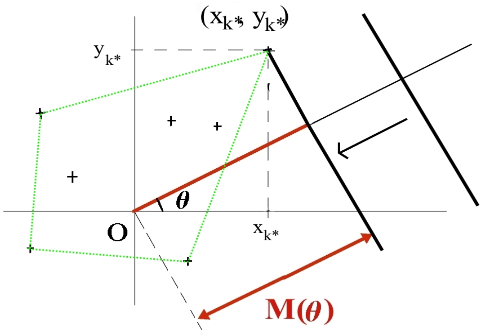
Cauchy's formulae Applied to Convex Polygon



$(x_k, y_k) \Longrightarrow$ vertices of the walk

$C \rightarrow$ Convex Hull with coordinates $\{X(s), Y(s)\}$ on C

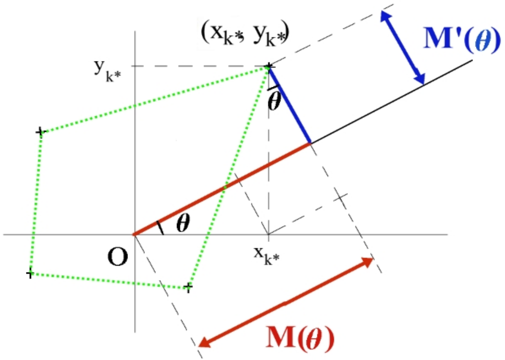
Cauchy's formulae Applied to Convex Polygon



$$M(\theta) = \max_{s \in C} [X(s) \cos\theta + Y(s) \sin\theta]$$
$$= \max_{k \in I} [x_k \cos\theta + y_k \sin\theta]$$
$$= x_{k^*} \cos\theta + y_{k^*} \sin\theta$$

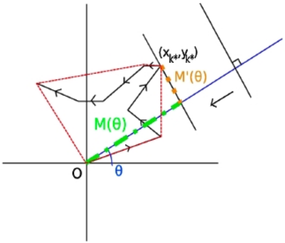
$k^* \rightarrow$ label of the point with largest projection along θ

Support Function of a Convex Hull



$$
\begin{aligned}
M(\theta) &= x_{k^*} \cos\theta + y_{k^*} \sin\theta \\
M'(\theta) &= -x_{k^*} \sin\theta + y_{k^*} \cos\theta
\end{aligned}
$$

Cauchy's Formulae Applied to Random Convex Hull



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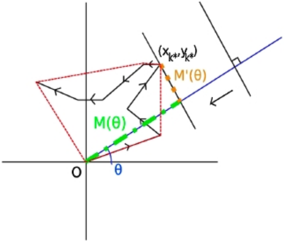


Mean perimeter of a random convex polygon

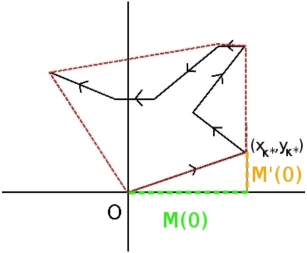
$$\langle L \rangle = \int_0^{2\pi} d\theta \, \langle M(\theta) \rangle$$

with $M(\theta) = x_{k^*} \cos\theta + y_{k^*} \sin\theta$

Mean area of a random convex polygon

$$\langle A \rangle = \frac{1}{2} \int_0^{2\pi} d\theta \left[\langle M^2(\theta) \rangle - \langle [M'(\theta)]^2 \rangle \right]$$

with $M'(\theta) = -x_{k^*} \sin\theta + y_{k^*} \cos\theta$

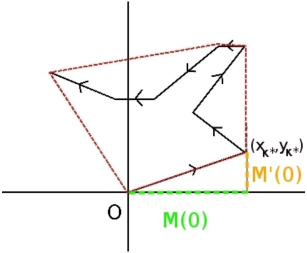


Mean Perimeter

$$\langle L \rangle = 2\pi \langle M(0) \rangle$$

with $M(\theta = 0) = \max_{k \in I} \{x_k\} = x_{k^*}$

Isotropically Distributed Points



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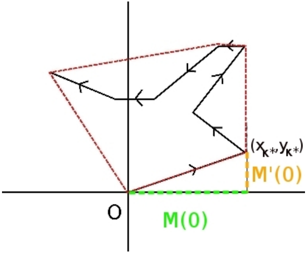
Mean Area

$$\langle A \rangle = \pi \left[\langle M^2(0) \rangle - \langle [M'(0)]^2 \rangle \right]$$

with $M'(\theta = 0) = y_{k^*}$

Isotropically Distributed Points



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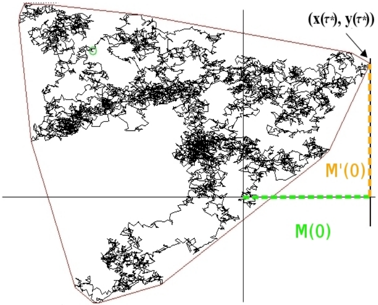
with $M'(\theta = 0) = y_{k^*}$

$\Longrightarrow$ Link to Extreme Value Statistics

Cauchy's Formulae Applied to the Convex Hull of a Brownian Path (n=1)



$x(\tau)$, $y(\tau)$ $\rightarrow$ a pair of independent one-dimensional Brownian motions: $0 \le \tau \le T$

$\frac{dx}{d\tau} = \eta_x(\tau)$
$\frac{dy}{d\tau} = \eta_y(\tau)$

Mean Perimeter

$$\langle L \rangle = 2\pi \langle M(0) \rangle$$

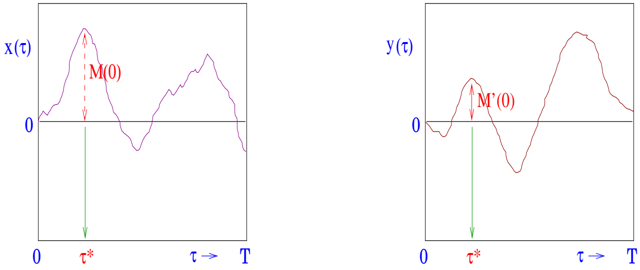
with $M(0) = \max_{0 \le \tau \le T} \{x(\tau)\} \equiv x(\tau^*)$

Mean Area

$$\langle A \rangle = \pi \left[\langle M^2(0) \rangle - \langle [M'(0)]^2 \rangle \right]$$

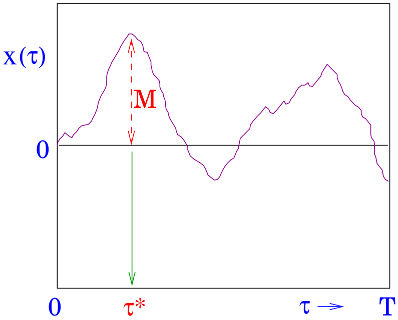
with $M'(0) = y(\tau^*)$

$M'(0) \to$ value of y at the special time τ^* when $x(\tau)$ is maximal



- $\langle M(0)\rangle = \int_0^\infty dM\ M\ \sigma_1(M|T)$; $\langle M^2(0)\rangle = \int_0^\infty dM\ M^2\ \sigma_1(M|T)$
- $\sigma_1(M|T) \to$ prob. density of maximum $M(0)$ of $x(\tau)$ in $[0,T]$
- $\langle [M'(0)]^2\rangle = \int_0^T d\tau^*\ \rho_1(\tau^*|T)\ \langle y^2(\tau^*)\rangle = 2D\langle \tau^*\rangle$

Distribution of M and τ^* for a single Brownian Path



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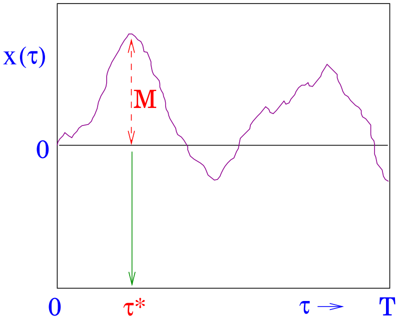
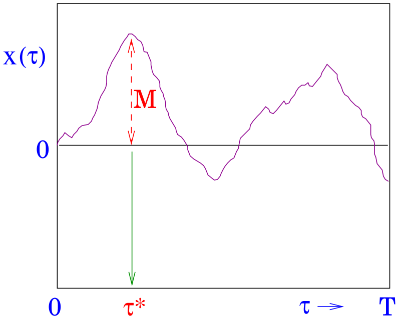


- Joint Distribution: $P_1(M, \tau^* | T) = \frac{M}{\pi {\tau^*}^{3/2} \sqrt{T - \tau^*}} \, e^{-M^2/2\tau^*}$ $\quad (D = 1/2)$

Distribution of M and τ^* for a single Brownian Path



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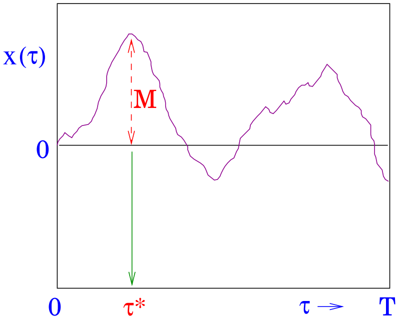
- Marginals: $\sigma_1(M|T) = \sqrt{\frac{2}{\pi T}}\, e^{-M^2/2T}$

Distribution of M and τ^* for a single Brownian Path



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- Marginals: $\sigma_1(M|T) = \sqrt{\frac{2}{\pi T}}\, e^{-M^2/2T}$

$$\rho_1(\tau^*|T) = \frac{1}{\pi\sqrt{\tau^*(T-\tau^*)}}$$ → Lévy's arcsine law

Distribution of the time τ^* at which a Brownian Motion is maximal over [0,T]

Lévy's Arcsine Law: $\rho_1(\tau^*|T) = \frac{1}{T}\ f_1\left(\frac{\tau^*}{T}\right)$

$$f_1(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

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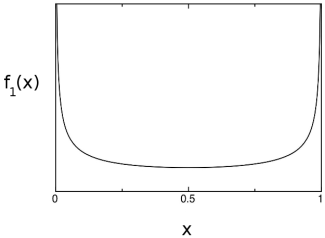
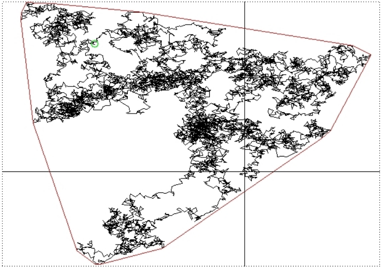
$$f_1(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

Cumulative distribution: $\mathrm{Prob}(\tau^* \leq t|T) = \frac{2}{\pi}\,\mathrm{arcsin}\left(\sqrt{t}\right)$



Results for n=1 Open Brownian Path



$x(\tau)$, $y(\tau)$ $\to$ a pair of independent one-dimensional Brownian motions over $0 \leq \tau \leq T$

Mean Perimeter

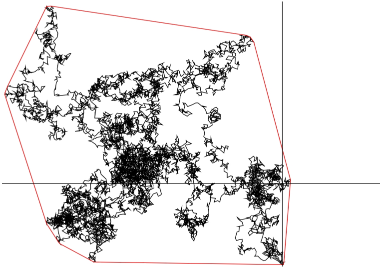
$$\langle L \rangle = \sqrt{8\pi T}$$

Mean Area

$$\langle A \rangle = \frac{\pi T}{2}$$

Takács, *Expected perimeter length*, Amer. Math. Month., **87** (1980)
El Bachir, (1983)

Results for n=1 Closed Brownian Path

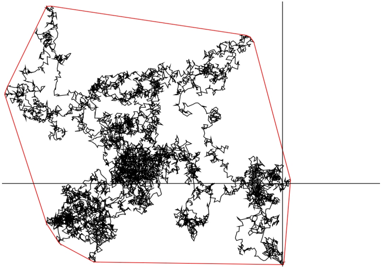


$x(\tau)$, $y(\tau)$ $\to$ a pair of independent one-dimensional Brownian bridges over $0 \leq \tau \leq T$

Mean Perimeter

$\langle L \rangle = \sqrt{\frac{\pi^3 T}{2}}$ Goldman, '96

Results for n=1 Closed Brownian Path



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Mean Perimeter

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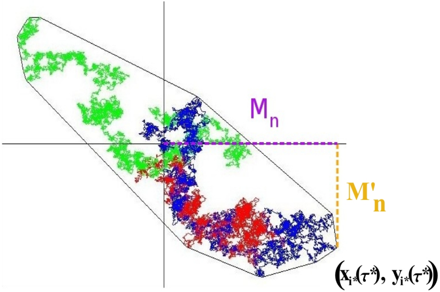
Mean Area

$$\boxed{\langle A \rangle = \frac{\pi T}{3}}$$ $\rightarrow$ New Result

Convex Hull of n Independent Brownian Paths



$x_i(\tau)$, $y_i(\tau)$ $\to$ $2\,n$ independent one-dimensional Brownian paths each of duration T

Mean Perimeter

$\langle L_n \rangle = 2\pi \langle M_n \rangle$

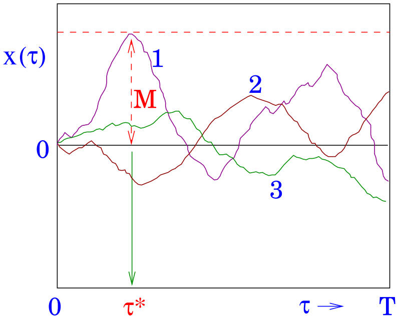
with $M_n = \max_{\tau,i} \{x_i(\tau)\} \equiv x_{i^*}(\tau^*)$

Mean Area

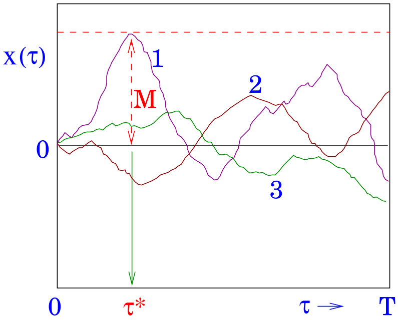
$\langle A_n \rangle = \pi \left[\langle M_n^2 \rangle - \langle [M_n']^2 \rangle \right]$

with $M_n' = y_{i^*}(\tau^*)$

Distribution of the global maximum M and τ^* for n paths



Distribution of the global maximum M and τ^* for n paths



- Joint Distribution: $P_n(M, \tau^*|T) = n\, P_1(M, \tau^*|T)\, \left[\mathrm{erf}\left(\frac{M}{\sqrt{2T}}\right)\right]^{n-1}$

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z du\; e^{-u^2}$$

Marginals of M and τ^* for arbitrary n

- Marginals: $\sigma_n(M|T) = \sqrt{\frac{2}{\pi T}}\, n\, e^{-M^2/2T} \left[\mathrm{erf}\left(\frac{M}{\sqrt{2T}}\right)\right]^{n-1}$

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$$\rho_n(\tau^*|T) = \frac{1}{T}\, f_n\,(\tau^*/T)$$

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$$\rho_n(\tau^*|T) = \frac{1}{T}\, f_n\left(\tau^*/T\right)$$

$$f_n(x) = \frac{2n}{\pi\sqrt{x(1-x)}} \int_0^\infty u\; e^{-u^2}\; [\mathrm{erf}(u\sqrt{x})]^{n-1}\; du$$

Marginals of M and τ^* for arbitrary n

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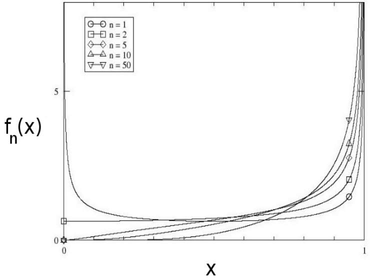
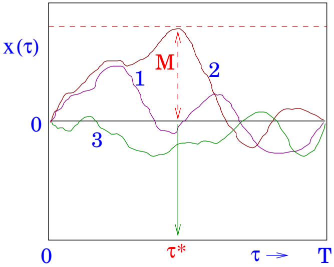
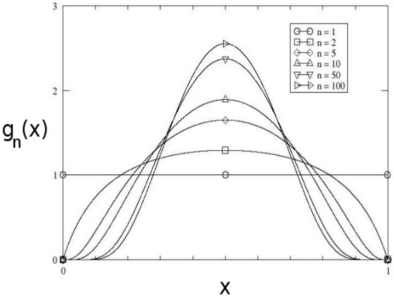
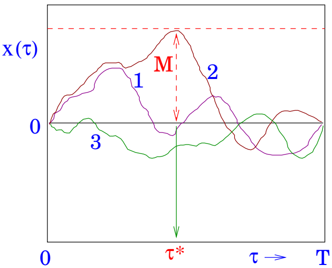


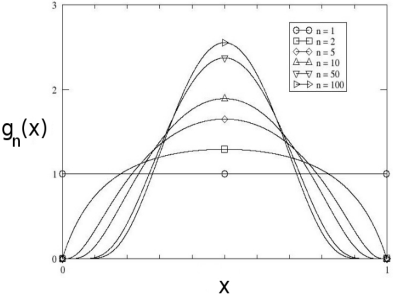
Marginals for n Independent Brownian Bridges





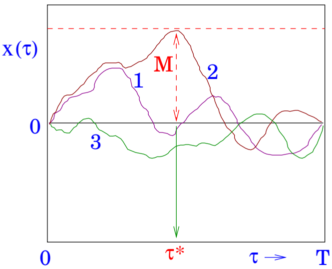
Marginals for n Independent Brownian Bridges

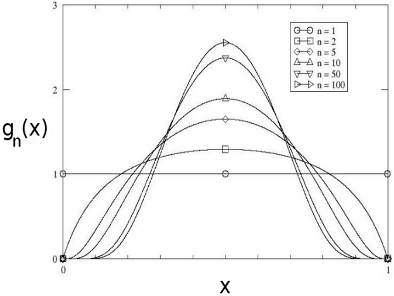




- Marginals: $\sigma_n(M|T) = \frac{4n}{T}\, M \left(1 - e^{-2M^2/T}\right)^{n-1}$

Marginals for n Independent Brownian Bridges





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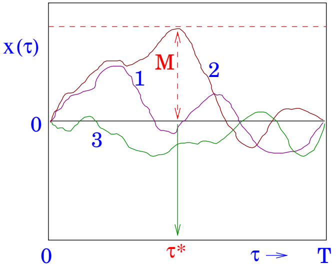
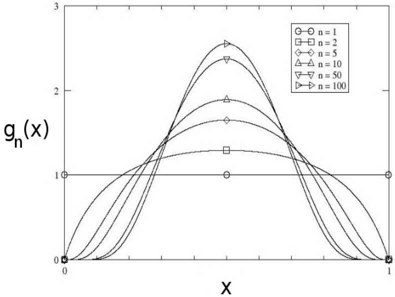
$$\rho_n(\tau^*|T) = \frac{1}{T}\, g_n(\tau^*/T)$$

Marginals for n Independent Brownian Bridges





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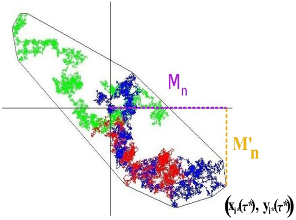
$$\rho_n(\tau^*|T) = \frac{1}{T}\, g_n\left(\tau^*/T\right)$$

$$g_n(x) = n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(-1)^k}{[1+4\,k\,x\,(1-x)]^{\frac{3}{2}}}$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



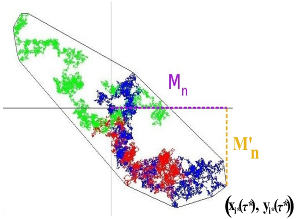
$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian paths over $0 \leq \tau \leq T$

Mean Perimeter (open paths)

$$\langle L_n \rangle = 2\pi \langle M_n \rangle$$

with $M_n = \max_{\tau,i} \{x_i(\tau)\} \equiv x_{i^*}(\tau^*)$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian paths over $0 \le \tau \le T$

Mean Perimeter (open paths)

$$\langle L_n \rangle = \alpha_n \sqrt{T}$$

$$\alpha_n = 4n\sqrt{2\pi} \int_0^\infty du\ u\ e^{-u^2}\ [\mathrm{erf}(u)]^{n-1}$$

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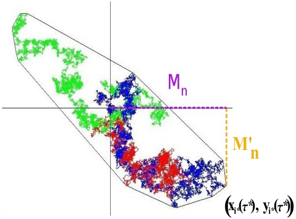

$x_i(\tau)$, $y_i(\tau)$ $\rightarrow$ $2\,n$ independent one-dimensional Brownian paths over $0 \leq \tau \leq T$

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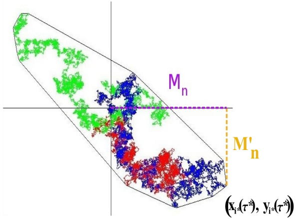
$$\alpha_1 = \sqrt{8\pi} = 5,013..$$

$$\alpha_2 = 4\sqrt{\pi} = 7,089..$$

$$\alpha_3 = 24\,\frac{\tan^{-1}\left(1/\sqrt{2}\right)}{\sqrt{\pi}} = 8,333..$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



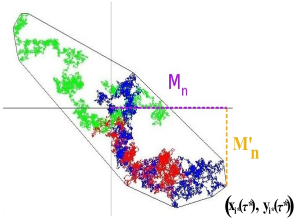
$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian paths over $0 \leq \tau \leq T$

Mean Area (open paths)

$$\langle A_n \rangle = \pi \left[\langle M_n^2 \rangle - \langle [M_n']^2 \rangle \right]$$

with $M_n' = y_{i^*}(\tau^*)$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Open Brownian Paths



$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian paths over $0 \le \tau \le T$

Mean Area (open paths)

$\langle A_n \rangle = \beta_n T$

$$\beta_n = 4n\sqrt{\pi} \int_0^\infty du \;\, u \;\, [\mathrm{erf}(u)]^{n-1} \left(u e^{-u^2} - h(u)\right)$$

$$h(u) = \frac{1}{2\sqrt{\pi}} \int_0^1 \frac{e^{-u^2/t}\, dt}{\sqrt{t(1-t)}}$$

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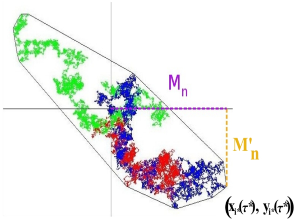

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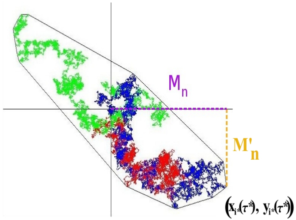
$$\beta_1 = \frac{\pi}{2} = 1,570..$$

$$\beta_2 = \pi = 3,141..$$

$$\beta_3 = \pi + 3 - \sqrt{3} = 4,409..$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Closed Brownian Paths



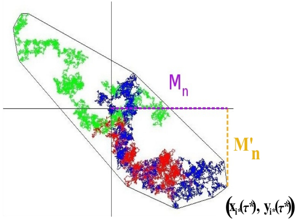
$x_i(\tau)$, $y_i(\tau) \to 2\,n$ independent one-dimensional Brownian bridges over $0 \le \tau \le T$

Mean Perimeter (Closed Paths)

$\langle L_n^c \rangle = \alpha_n^c \sqrt{T}$

$$\alpha_n^c = \frac{\pi^{3/2}}{\sqrt{2}} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{\sqrt{k}}$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Closed Brownian Paths



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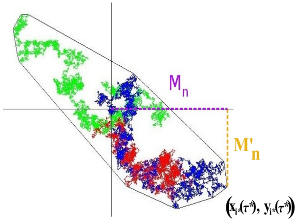
$$\begin{aligned} \alpha_1^c &= \sqrt{\pi^3/2} = 3,937. \\ \alpha_2^c &= \sqrt{\pi^3}(\sqrt{2} - 1/2) = 5,090.. \\ \alpha_3^c &= \sqrt{\pi^3}\left(\frac{3}{\sqrt{2}} - \frac{3}{2} + \frac{1}{\sqrt{6}}\right) = 5,732.. \end{aligned}$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Closed Brownian Paths



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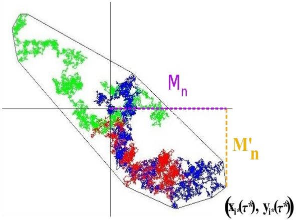
$\langle A_n^c \rangle = \beta_n^c\, T$

$$\beta_n^c = \frac{\pi}{2}\left[\sum_{k=1}^{n}\frac{1}{k} - \frac{n}{3} + \frac{1}{2}\sum_{k=2}^{n}(-1)^k\, w(k)\right]$$

$$w(k) = \binom{n}{k}\,(k-1)^{-3/2}\,\left(k\tan^{-1}(\sqrt{k-1}) - \sqrt{k-1}\right)$$

Mean Perimeter and Mean Area of the Convex Hull of n Independent Closed Brownian Paths



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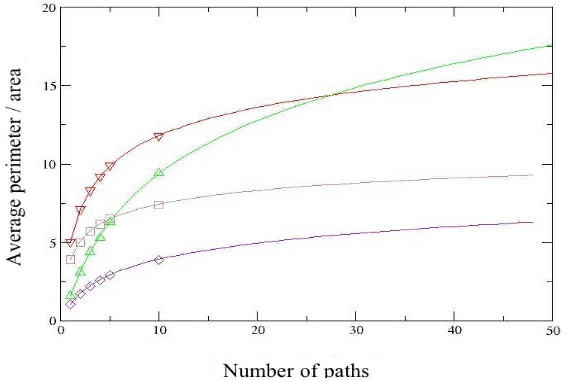
$$\beta_1^c = \frac{\pi}{3} = 1,047..$$

$$\beta_2^c = \frac{\pi(4+3\pi)}{24} = 1,757..$$

$$\beta_3^c = 2,250..$$

Numerical Check



The coefficients α_n (mean perimeter) (lower triangle), β_n (mean area) (upper triangle) of n open paths and similarly α_n^c (square) and β_n^c (diamond) for n closed paths, plotted against n. The symbols denote numerical simulations (up to $n = 10$, with 10^3 realisations for each point)

Asymptotics for large n

For n open paths:

$$\langle L_n \rangle \simeq \left(2\,\pi\sqrt{2\ln n}\right)\ \sqrt{T}$$

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- Very slow growth with $n \implies$ good news for conservation

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- Unified approach adapting Cauchy's formulae

  $\Longrightarrow$ Mean Perimeter and Area of Random Convex Hull

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  $\Longrightarrow$ Ecological Implication: Home Range Estimate

Very slow (logarithmic) growth of Home Range with population size n

Open Questions

- For n planar random walks each of N steps

  Mean no. of Vertices $\langle V_n(N) \rangle \to$?

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- Non-Brownian paths $\to$ anomalous diffusion, e.g., Lévy flights, external potential ?

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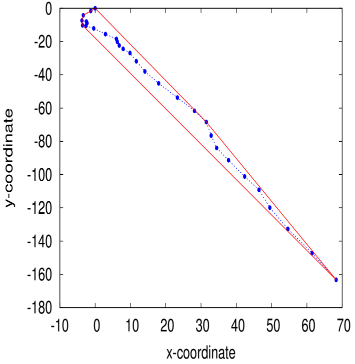

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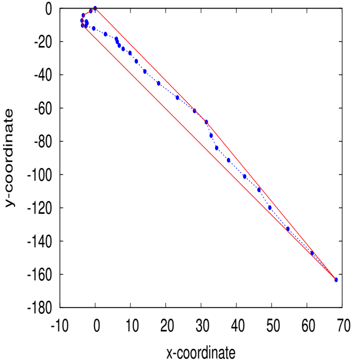

- Exact results for the mean perimeter and mean area

mean perimeter:

$$\langle L_1 \rangle = \frac{3\pi}{2}\, T^{3/2}$$

mean area:

$$\langle A_1 \rangle = \frac{5\pi}{192}\sqrt{\frac{3}{2}}\, T^3$$

A. Reymbaut, S.M. and A. Rosso (arXiv: 1108.5455), to appear in J. Phys. A: Math. Theor. (2011)