Hall effect, Jack polynomials, CFT…..(More to come in the talk by D. Serban)

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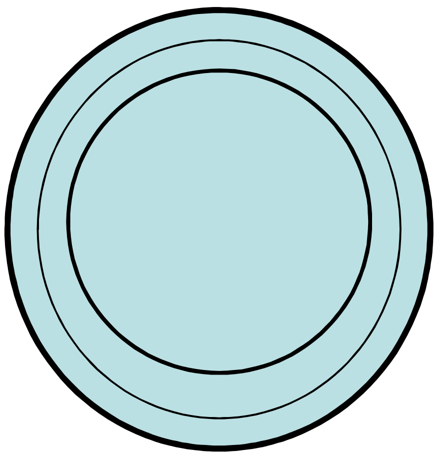
Most striking occurrence of a quantum macroscopic effect in the real word

- In presence of of a magnetic field, the conductivity is quantized to be a simple fraction up to 10^{-10}
- Important experimental fractions for electrons are: $$\frac{p}{2p+1}$$
- Theorists believe for bosons: $$\frac{p}{p+1}$$
- Fraction called filling factor.

Hall effect

- Lowest Landau Level wave functions

$$\psi_n(z) = \frac{z^n}{n!} e^{-\frac{z\bar{z}}{l^2}}$$



$$r = l\sqrt{n}$$

n=1,2, …, $\frac{A}{2\pi l^2}$

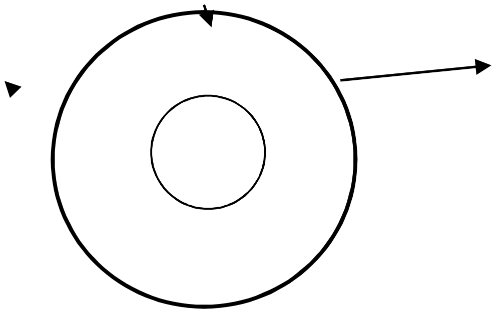
$$\frac{A}{2\pi l^2} = n_0$$

Number of available cells also
the **maximal degree** in each variable

$$S(z_1^{\lambda_1} \dots z_n^{\lambda_n}) = m_\lambda$$

Is a basis of **states** for the system
Labeled by partitions

$$AS(z_1^{\lambda_1} \dots z_n^{\lambda_n}) = S_\lambda \Delta$$

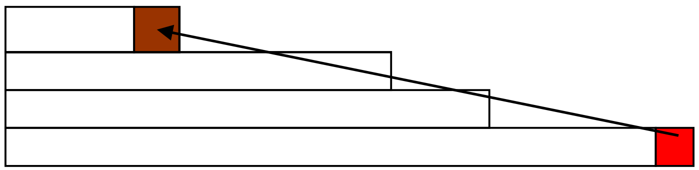


N_k particles in the orbital k are the occupation numbers equivalently,the number of times the row of length k occurs in the partition.

$$N_0 N_1 N_2 \dots . N_k \dots ..$$

$$z_1^0 \dots z_{N_0}^0 z_{N_0+1}^1 \dots z_{N_0+N_1}^1 \dots .$$

Can be represented by a partition with N_0 particles in orbital 0, N_1 particles in orbital 1…N_k particles in orbital k…..



There exists a partial order on partitions, the squeezing order

Interactions translate into repulsion between particles.

$$\left(z_i - z_j\right)^m$$

m universal measures the strength of the interactions.

Competition between interactions which spread electrons apart and high compression which minimizes the degree n. Ground state is the minimal degree symmetric polynomial compatible with the repulsive interaction.

Laughlin wave function occupation numbers:

Keeping only the dominant weight of the expansion

$$0010010010 0$$

1 particle at most into m orbitals (m=3 here).

Important quantum number with a topological interpretation

- Filling factor equal to number of particles per unit cell:
- $$\nu = \frac{\textbf{Number of variables}}{\textbf{Degree of polynomial}}$$

$$\frac{1}{3}$$ **In the preceding case.**

Excitation

$$\prod_i (x - z_i) \prod_{i<j} (z_i - z_j)^m$$

Imagine you put the excitation at x=0

$$0001001001000$$

It is pushing the motive to the right, inserting a 1/3 charge

Edge Physics

- We shall make a not very well established assumption that electrons at the edge of the sample are relevant.
- Edge state physics described by a 1+1dimentional model.

Non commutativity.

- The dynamical degrees of freedom (r,p) are reorganized into two commuting sets, the dynamical momenta and guiding centers which obey heisenberg algebra:

$$\rho_p \rho_q = e^{ip\times q l^2} \rho_{p+q}$$

where l is the magnetic length, p takes N^2 values.

algebra realized on edge states as:

$$W_{1+\infty}$$

$$[\rho_p, \rho_q] =$$

$$(e^{ip\times ql^2} - e^{-ip\times ql^2})\rho_{p+q} + c\delta_{p+q}$$

Jack Polynomials:

$$L_0 = \sum_{i=1}^{n} z_i \partial_{z_i}$$

$$H^g = \sum_{i=1}^{n} (z_i \partial_{z_i})^2 + g \sum_{i \neq j} \frac{z_i + z_j}{z_i - z_j} z_i \partial_{z_i}$$

Jack polynomials are eigenstates of the Calogero-Sutherland Hamiltonian on a circle with 1/r^2 potential interaction.

Remarks

- By conjugating with a Vandermonde determinant, $\Delta = \prod_{i<j} (z_i - z_j)^g$
- $g = m$ integer Hall effect
- H can be made Hermitian:

$$H = -\sum \partial^2{}_{\theta_i} + \frac{1}{4} \sum_{i<j} \frac{g(g-1)}{\sin(\theta_i - \theta_j)^2}$$

- In particular H is noninteracting if g=0,1
- Deformation of Schur functions.

Eigenstates

- C.S Hamiltonian is triangular in the monomial basis. Spectrum:

$$H' = H - NgL_0$$

$$E_\lambda^g = \sum_{i=1}^{N} \lambda_i(\lambda_i + g(1-2i))$$

Duality

$$b(\lambda) = \sum (i-1)\lambda_i = \sum \lambda'_i (\lambda'_i - 1)/2$$

$$E(\lambda) = b(\lambda') - g b(\lambda)$$

$$E_\lambda^g + g E_{\lambda'}^{1/g} = 0$$

Jack polynomials at

$$g = -\frac{r-1}{k+1}$$

Feigin-Jimbo-Miwa-Mukhin

Generate ideal of polynomials"vanishing as the r power of the Distance between particles (difference Between coordinates) as k+1 particles come together.

Exclusion statistics:

- No more than k particles into r consecutive orbitals. $\lambda_i - \lambda_{i+k} \geq r$

For example when

k=r=2, the possible ground states (most dense packings) are given by:

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Filling factor is $\boxed{\frac{r}{k}}$

CFT

- Parafermionc minimal models :

$$WA_{k-1}(k+1, k+r)$$

$$\psi_q \times \psi_p = \psi_{q+p}$$

- Models with Z_k symmetry (q=q+k) .
- r=2 (Fateev-Zamolodchikov 85). r=3,4 generealizations.

Clustering properties

- Due to chiral ring properties:

$$<\psi_1(z_1)\psi_1(z_2)\ldots\psi_1(z_N)> \prod_{i<j}(z_i - z_j)^{k/r}$$

- Is a polynomial vanishing with a degree r when k+1 variables are clustered. This indicates a possible connection between Jack polynomials and parafermionic CFT.

Moore Read (k=r=2)

When **3 electrons** are put together, the wave function vanishes as: ε^2

$$\mathrm{Pfaff}\left[\frac{1}{z_i - z_j}\right]\prod_{i<j}(z_i - z_j)$$

Annulations

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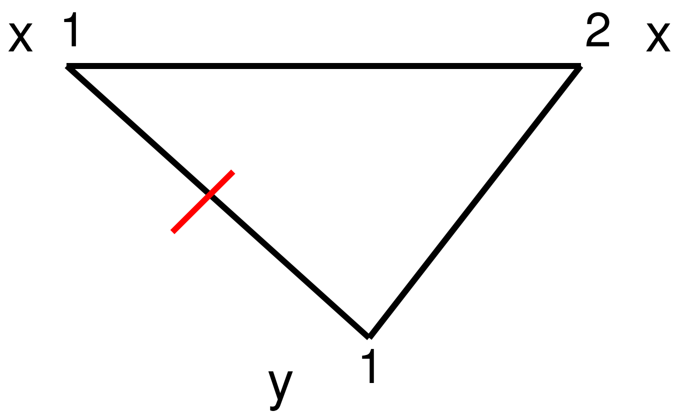
When **3 electrons** are put together, the wave function vanishes as: ε^2



Fractional excitations

$$\prod (x - z_i) Pf\left(\frac{1}{z_i - z_j}\right)$$

Can be split into 2 charge ½ excitations:

$$Pf\left(\frac{(x - z_i)(y - z_j) + x \leftrightarrow y}{z_i - z_j}\right)$$

CFT analogous:

- Quasiparticles are represented by Ising field.
- Jastrow factor necessary to insure locality between electrons and quasiparticles.
- Quasiparicles obey braid statistics.

$$< \sigma(z_1)\sigma(z_2)....\psi(w_1)....\psi(w_N) > \prod (z_i - w_j)^{1/2}$$

Jack polynomials=CFT correlators and more?

- q deformed case.
- Bosonic Hamiltonian from CS.
- Duality.
- Another derivation from nullvector.
- More Hamiltonians and representations, some related to AGT conjecture.(Didina Serban talk)

q deformed CS=Macdonald

- Very interesting deformation of the CS model Ruisjenaars Hamiltonian (also Macdonald):

$$H = \sum_i \prod_{i \neq j} \frac{tz_i - z_j}{z_i - z_j} T_{z_i \to qz_i}$$

- 2 parameters q,t >g in limt q=t=1.
- Also H' obtained t,q<>1/t,1/q.

Hecke Algebra (Lascoux Schutzenberger)

$$T_{12}\Psi = (\Psi(z_1, z_2) - \Psi(z_2, z_1))\frac{z_1 t - z_2 t^{-1}}{z_1 - z_2}$$

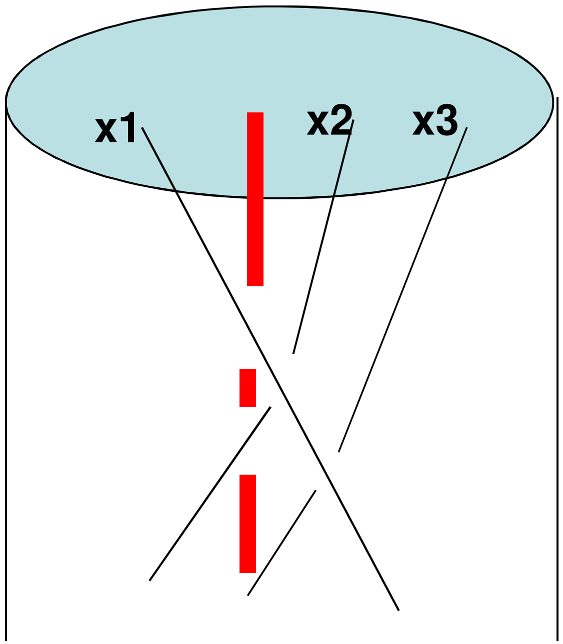
T projects onto polynomials divisible by:

$$z_1 t - z_2 t^{-1}$$

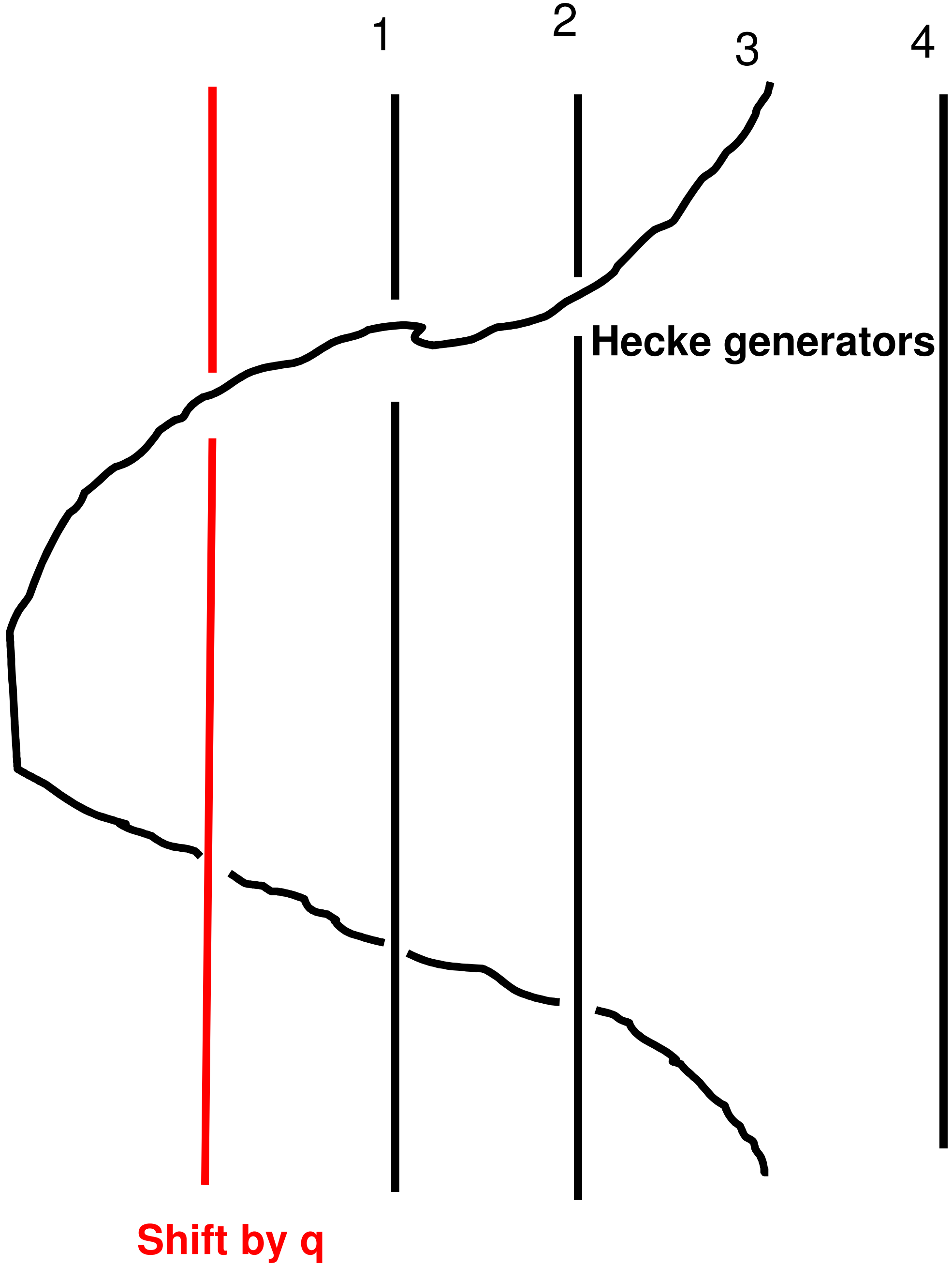
T obeys Braid group algebra relations

With adiabatic time QHE=TQFT

One must consider the space P(x1,x2,x3|zi) of polynomials vanishing when zi-xj goes to 0.



Compute Feynman path integrals



Y_3

$$[Y_i, Y_j] = 0$$

Collective variables

- Jevicki Sakita (NPB 165,1980)
- Stanley (Adv. Math. 77, 1989)
- Consider the action of the Hamiltonian on a wave function depending on collective variables:

$$\Psi(p_1, p_2, \ldots)$$

with

$$p_k = \sum z_i^k$$

Collective Hamiltonian:

- H=K+gV

$$= \sum_{n} n^2 p_n \partial_n + \sum_{n,m} p_{n+m} n\partial_n m\partial_m$$

$$+ g\left(\sum_{k+l=n} p_k p_l + (N-1) p_n\right) n\partial_n$$

In terms of bosons:

$$a_{-n} = g^{1/2} p_n$$

$$a_n = g^{-1/2} n \partial_n$$

$$H = (1 - g) \sum_{n>1} a_{-n} a_n + g^{1/2}/3 \sum_{n+m+l=0} a_n a_m a_l + NgL_0$$

In terms of Virasoro

- H is hermitian but p depend on g, therefore,

$$< p_n, p_m >= \frac{n}{g} \delta_{n,m}$$

- Macdonald approach Hilbert Schmidt.

- H= $\sum_{n>0} a_{-n} l_n$ (Awata et al.)

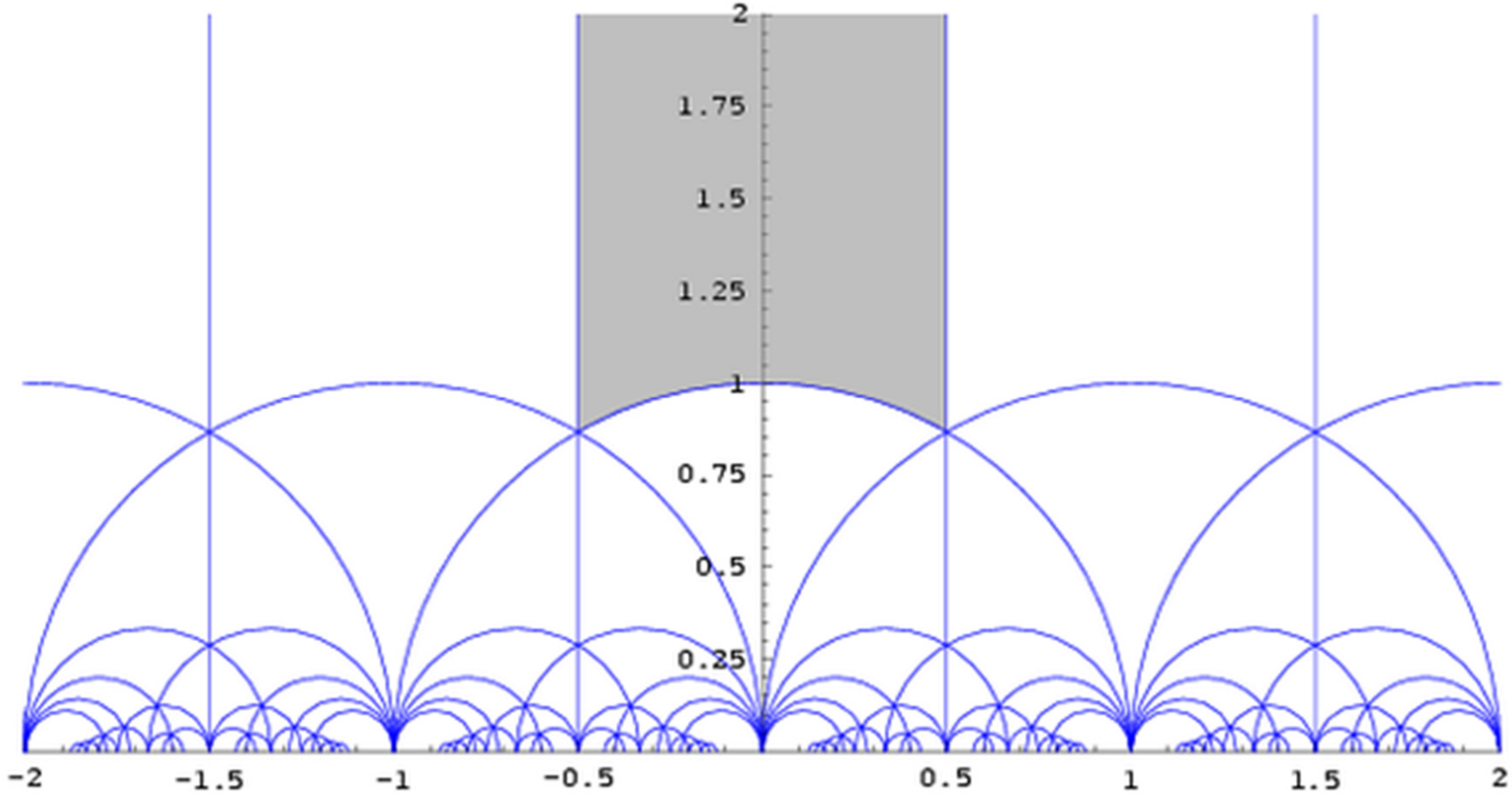
Virasoro Algebra constructed via FF construction

$$c = 13 - 6(g + \frac{1}{g})$$

q-deformed case.

- Hamiltonians become zero mode of vertex operator analogous to e,f generators of SL2(q). Ding Iohara (Shiraishi et al.)
- Can be thougt as (central extension of) algebra generated by: $\sum y_i^{\,k}, \sum z_i^{\,k}$
- Analogous of Cartan generators K generated by: $\sum z_i$
- Also extention of : $W_{1+\infty}$

SL2(Z) symmetry.



Classical limit

- Jevicki (81). Abanov Bettelheim, Wiegmann (08).
- Set

$$\partial \varphi = \sum_{n>0} a_n z^{-(n+1)} = v / \sqrt{g}$$

- Then:

$$H = \int \frac{1}{3} v^3 + \frac{1}{2} v . h(v_x) dx$$

Duality again:

- Recall the way quasiparticles are constructed in the Hall effect:

$$\prod_i (x - z_i) \prod_{i<j} \left(z_i - z_j\right)^m$$

- So, we can consider the Kernel:

$$\prod_{a,i} (x_a - z_j)$$

Kernel as intertwiner

- This Kernel intertwines electrons and quasiholes with fractional charge 1/g (Laughlin argument and g an integer),
- It is not difficult to show (Macdonald,Gaudin):

$$H_x^{1/g} + H_z^{g} = 0$$

Correlators?

- Derive this Hamiltonian by using Correlation functions in CFT. (Cardy…) Consider degenerate fields (g=-1/3 Moore Read) corresponds to :

$$V_g = e^{-(g/2)^{1/2}\varphi}$$

- Obeying level 2 degeneracy equation:

$$\partial_x{}^2 V_g(x) = gT(x)V_g(x)$$

Another expression:

- Going back to the boson representation for the Virasoro generators (F.F.), we can also rewrite the conserved quantities as a sum of two BO hamiltonians+ a coupling

  (Generalizing a formula of Belavin,Belavin)

$$I_3^+ = H_3^+(c) + H_3^+(c') + (1-g)\sum_{m>0} c_{-m} c'_m$$

Duality relation:

- The same can be repeated for the dual operators, absence of interations between the two type of fields implies similar separation as for the abelian case:

$$H_x^{1/g} + H_z^{g} = 0$$

- Implying a factorization of the conformal blocs:

$$F(x,z) = \sum_{\lambda} P_{\lambda}(x) P_{\lambda'}(z)$$

In the Ising case

- Correlator becomes factorized:

$$< \sigma(z_1)\sigma(z_2)....\psi(w_1)....\psi(w_N) > \prod (z_i - w_j)^{1/2}$$

- Jastrow factor can be taken into accout by U(1) factor.

$$V_g = e^{-(g/2)^{1/2}\varphi} \qquad V_{1/g} = e^{-(1/2g)^{1/2}\varphi}$$

Organization of states.

- Didina Serban will show that the Hamiltonian is the sum of 2 independant C.S. Hamiltonians up to a lower triangular interaction term. So the spectrum is characterized by 2 sets of quantum numbers. Agrees with the exclusion statistics selection rules.

Conclusions extensions

- Can be generalized to W-n theories in particular to parafermionic theories relevant for the Hall effect (Haldane Rezayi serie).
- CS Hamiltonian seems to provide a « good » bases to classify the descendants inside a Virasoro module.
- A lot more need to be understood, non polynomial solutions of C.S. (Didina's talk) , relevance for the QHE etc…