Return Probability for the Loop-Erasing Random Walk

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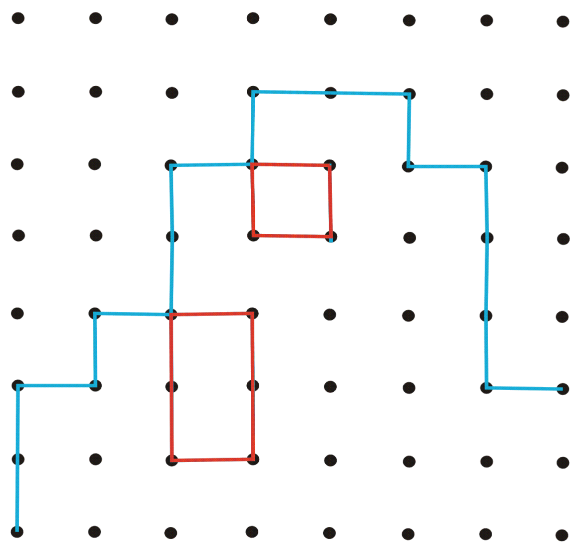
Loop erased random walk on square lattice



Lawler, 1980

Pemantle,1991

Majumdar,1992

Kenyon, 2000

Lawler, Schramm, Werner, 2004

LERW: a path obtained from the simple random walk by deleting all cycles in chronological order

1. The problem:

What is probability P(0,1) that the LERW starting from the origin (0,0) visits ever the neighboring point (0,1) ?

The conjecture: P(0,1)= 5/16

Poghosyan, V.P., 2010

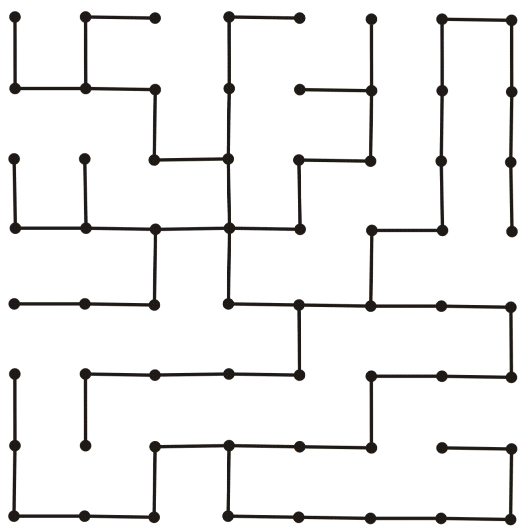
Levine, Peres, 2011

2. Related problem and conjecture:

Average height in the Abelian sandpile $<\eta>=25/8$.

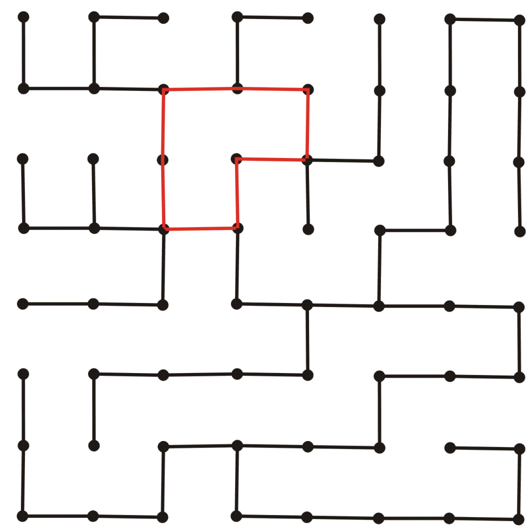
Grassberger, 1994

Spanning tree (V=E+1)



Number of spanning trees
det Δ^* (Kirhhoff, 1847)

Unicycle (V=E)



Number of unicycles - ?

Δ^* is the Laplacian where one diagonal element $\Delta^*_{ii} = \Delta_{ii} + 1$

Two more conjectures

1. In the limit of infinitely large lattice
(number of unicycles) / (number of spanning trees)
= 1/8

2. The average length of the cycle in unicycles is 8

Levine, Peres , 2011

All four conjectures are reduced to the first one (5/16).

The plan

1. Bijection between spanning trees and recurrent sandpile configurations.

2. Proof of the conjecture 5/16.

3. Further problems.

Abelian Sandpile model

An integer height z_i is ascribed to each site.
The evolution is defined by rules:

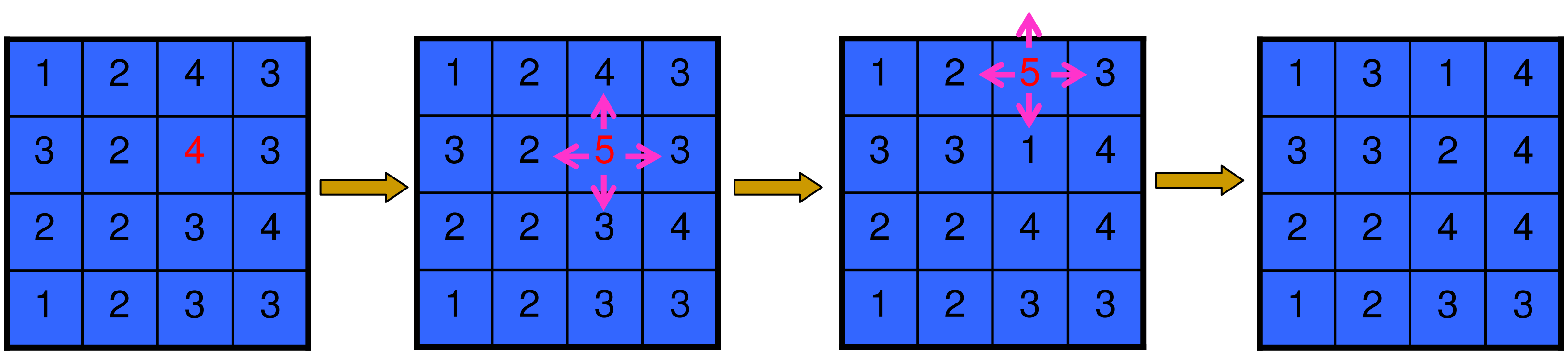
$$z_i \to z_i + 1 \qquad z_i = 1, 2, 3, 4$$

Bak, Tang, Wiesenfeld, 1987

If any $z_i \geq z_i^C = \deg(i)$

$$z_i \to z_i - z_i^C$$

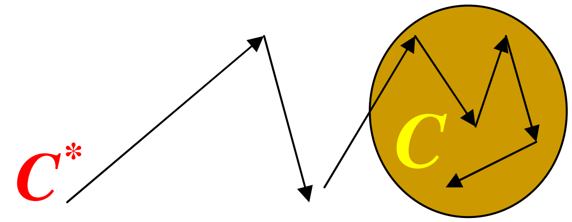
$$z_j \to z_j + 1 \qquad j \text{ is neighbor of } i$$



Allowed and Forbidden Configurations

Allowed configurations = Recurrent Configurations:

each configuration C is reachable from an arbitrary one C^* by sandpile dynamics



Forbidden configuration

D. Dhar, 1990

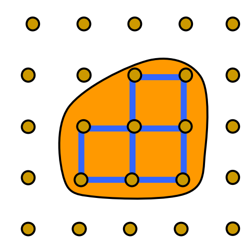
| 1 | 1 |

possible evolution: | 5 | 1 | → | 1 | 2 | → | 1 | 5 | → | 2 | 1 |

| 1 | 2 | 1 |

General FSC :

$$z_j \leq \text{ \# of nearest neighbors of } j \text{ in } \Gamma$$



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Mapping onto Spanning Trees

C – Sandpile configuration

$[\hat{a}_i, \hat{a}_j] = 0$

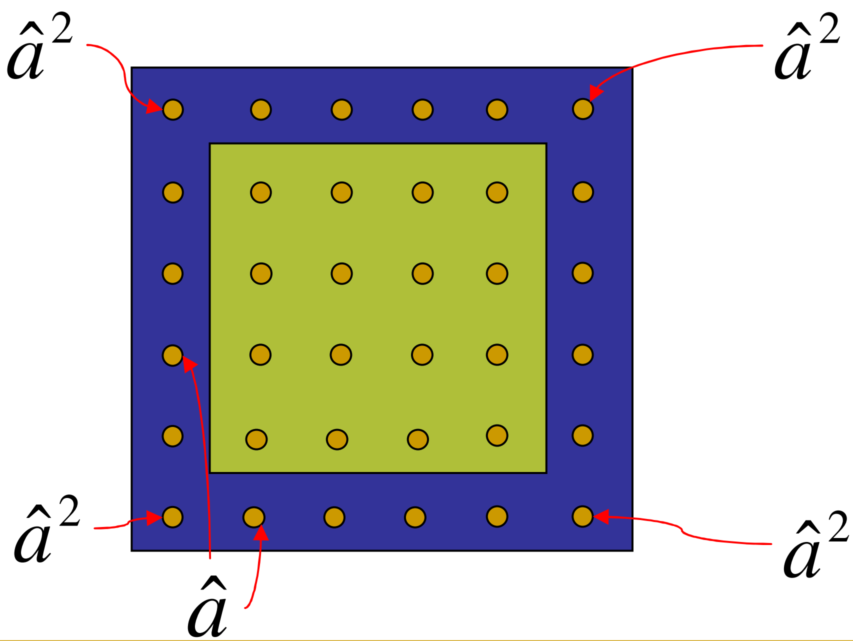
$\hat{a}_i C = C'$ - operator of adding a particle to site i

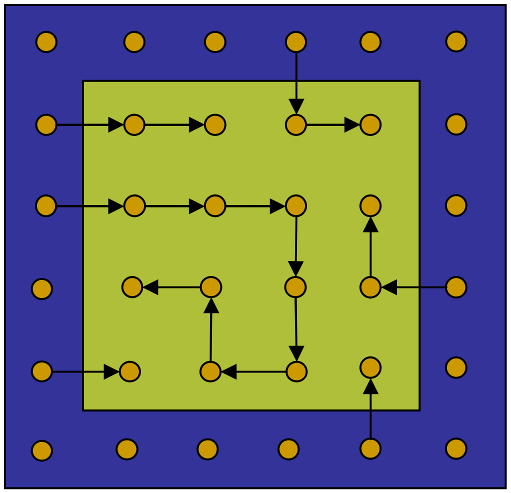
D.Dhar, PRL 64, 1613 (1990)

$\hat{a}_i^{-1}$ - inverse operator

Identity operator $\sum_j \hat{a}_i^{\Delta_{ij}} = 1$

Another form of identity operator





Height Probabilities

1	4	3	2	3
3	2	4	4	1
2	4	3	2	4
3	2	3	3	1
1	4	2	3	3

$z_i = 1, 2, 3, 4$ - local heights of recurrent sandpile

$$P_1 = Prob(z_i = 1) = \frac{2(\pi - 2)}{\pi^3}$$

S.Majumdar, D. Dhar, 1991

$$P_2 = \frac{1}{2} - \frac{3}{2\pi} - \frac{2}{\pi^2} + \frac{12}{\pi^3} + \frac{I_1}{4}$$

$$P_3 = \frac{1}{4} + \frac{3}{2\pi} + \frac{1}{\pi^2} - \frac{12}{\pi^3} - \frac{I_1}{2} - \frac{I_2}{32}$$

$$P_4 = 1 - P_1 - P_2 - P_3$$

V.P. (1994)

Evaluation of integrals to twelve decimals leads to conjecture (Jeng,Piroux,Ruelle ,2006)

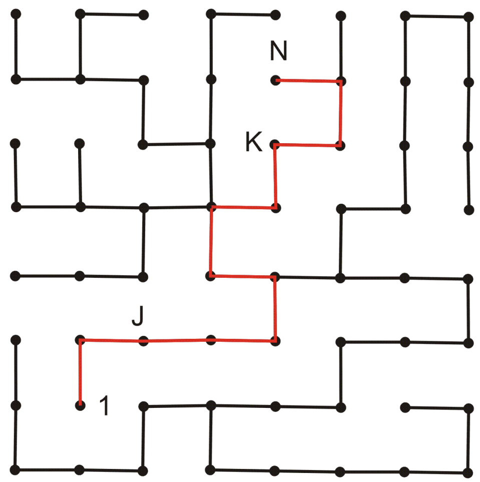
$$P_2 = \frac{1}{4} - \frac{1}{2\pi} - \frac{3}{\pi^2} + \frac{12}{\pi^3}$$

and

$$P_3 = \frac{3}{8} + \frac{1}{\pi} - \frac{12}{\pi^3}$$

$$< h >= P_1 + 2P_2 + 3P_3 + 4P_4 = 25/8$$

LERW and spanning trees



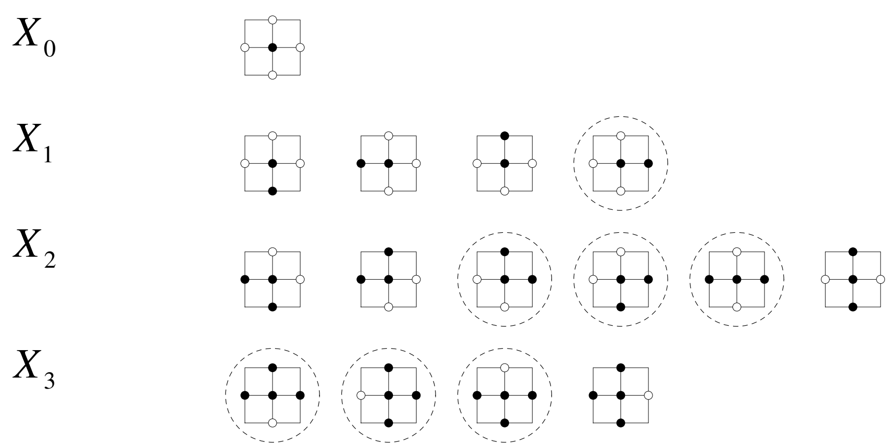
Red line is a path on the spanning tree = LERW.

Point j is called predecessor of point k on the LERW if the LERW from 1 to N passes j first.

Return probability P(0,1) is the probability that point (0,0) is the predecessor of point (0,1).

Predecessors and height probabilities (V.P., 1994)

Open circles are not predecessors of the central site



$$P_1 = \frac{X_0}{4N}, P_2 = P_1 + \frac{X_1}{3N}, P_3 = P_2 + \frac{X_2}{2N}, P_4 = P_3 + \frac{X_3}{N}$$

N is the total number of spanning trees

$$P(0,1) = \frac{X_1}{4N} + \frac{X_2}{2N} + \frac{3X_3}{4N}$$

$$P(0,1)=\frac{X_1}{4N}+\frac{X_2}{2N}+\frac{3X_3}{4N};$$ X_1, X_2, X_3 are irrational numbers

For instance, from the sandpile theory
$$\frac{X_1}{N}=\frac{3}{2}-\frac{9}{2\pi}-\frac{12}{\pi^2}+\frac{48}{\pi^3}+\frac{3I_1}{4}$$

where

$$I_1=\frac{1}{16\pi^4}\int\int\int\int_0^{2\pi}\frac{i\sin(\beta_1)\det(M_1)}{D(\alpha_1,\beta_1)D(\alpha_2,\beta_2)D(\alpha_1+\alpha_2,\beta_1+\beta_2)}d\alpha_1 d\alpha_2 d\beta_1 d\beta_2$$

with
$$D(\alpha,\beta)=2-\cos(\alpha)-\cos(\beta)$$

and
$$M_1=\begin{pmatrix} 1 & 1 & e^{i\alpha_2} & 1 \\ 3 & e^{i(\beta_1+\beta_2)} & e^{i(\alpha_2-\beta_2)} & e^{-i\beta_1} \\ 4/\pi-1 & e^{i(\alpha_1+\alpha_2)} & 1 & e^{-i\alpha_1} \\ 4/\pi-1 & e^{-i(\alpha_1+\alpha_2)} & e^{2i\alpha_2} & e^{i\alpha_1} \end{pmatrix}$$

(V.P.,1994)

Why is P(0,1) rational?

The idea of proof

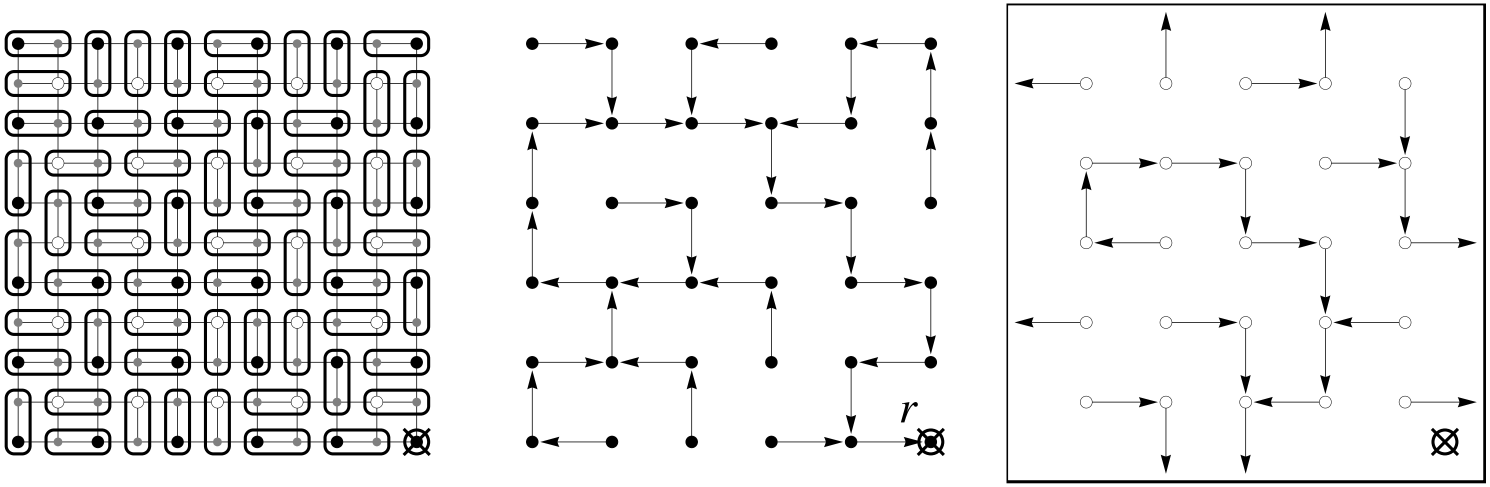
1. Temperley's correspondence between the dimer model and spanning trees

2. Monomer impurities as sinks of lattice paths on trees

3. Assembling the LERW from two lattice paths.

Then the problem is reduced to evaluation of the monomer-monomer correlation function.

Dimers and spanning trees

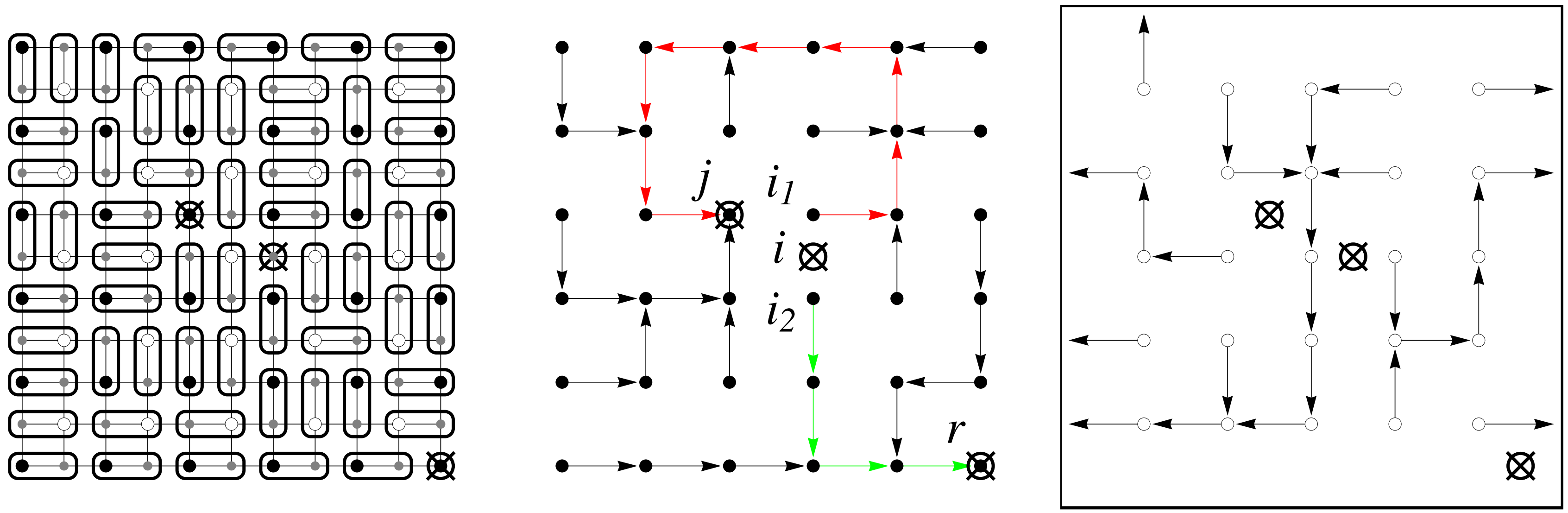


(a) (b) (c)

Black circles: odd-odd sublattice; white: even-even sublattice

r – the root of the spanning tree on the odd-odd sublattice

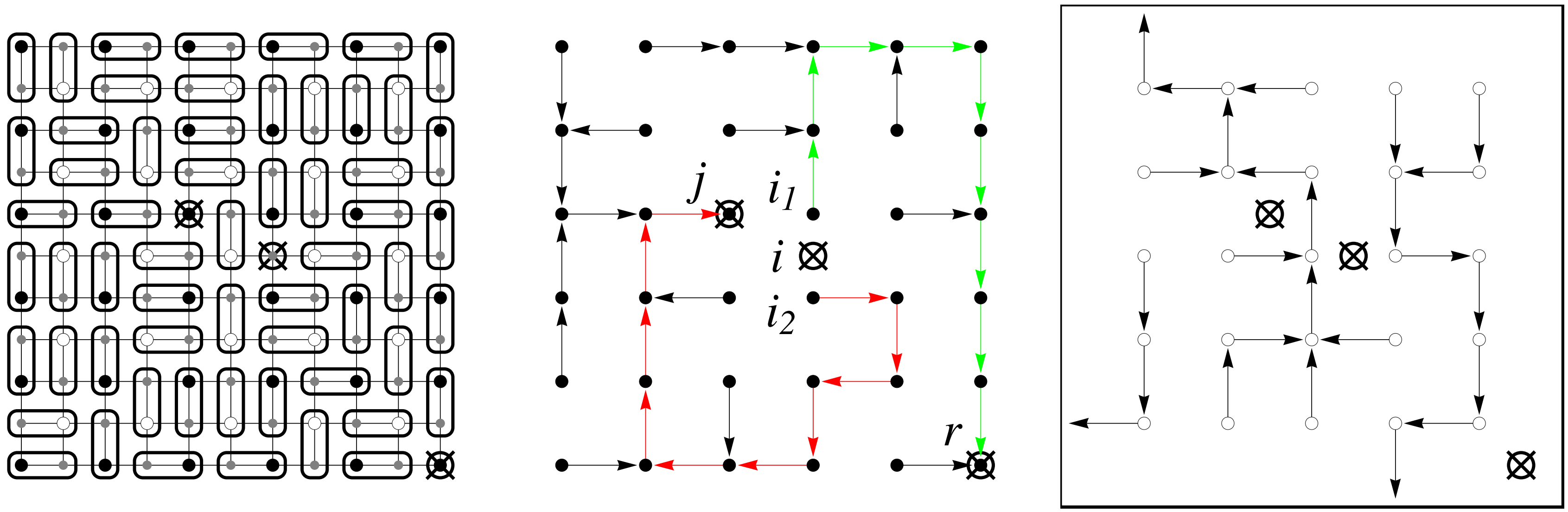
Dimers and two monomers (M_1)



(a) (b) (c)

(b) red path from i_1 to j ; green path from i_2 to the root.

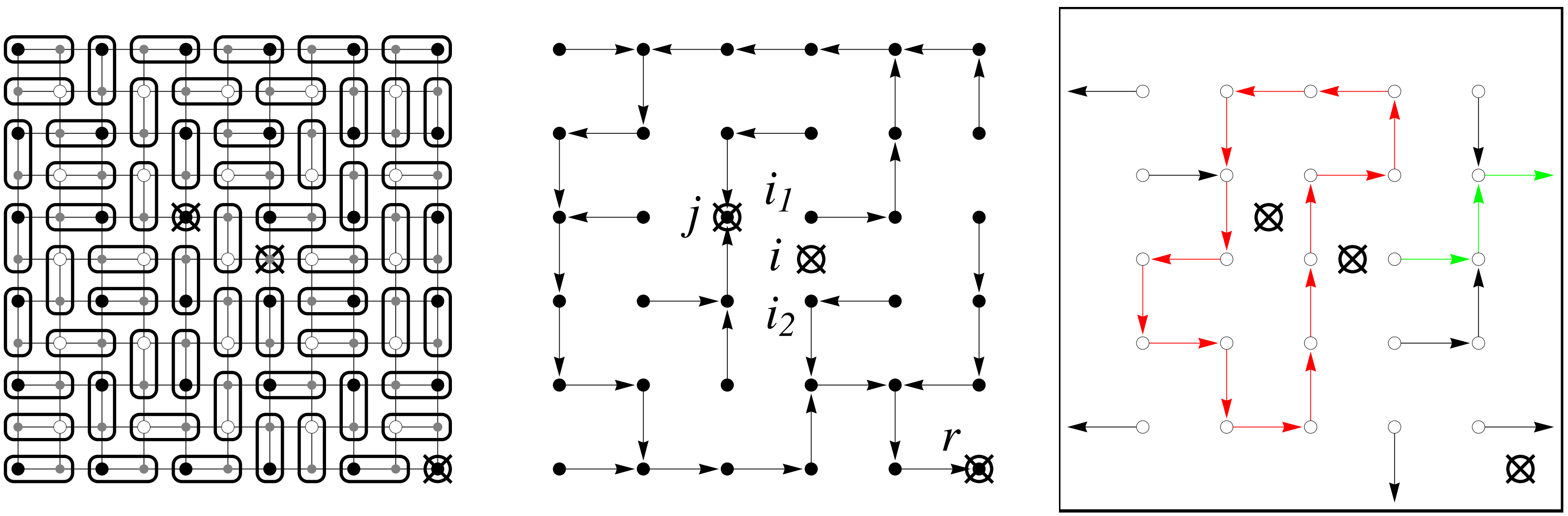
Dimers and two monomers (M_2)



(a) (b) (c)

(b) red path from i_2 to j ; green path from i_1 to the root.

Dimers and two monomers (M_3)

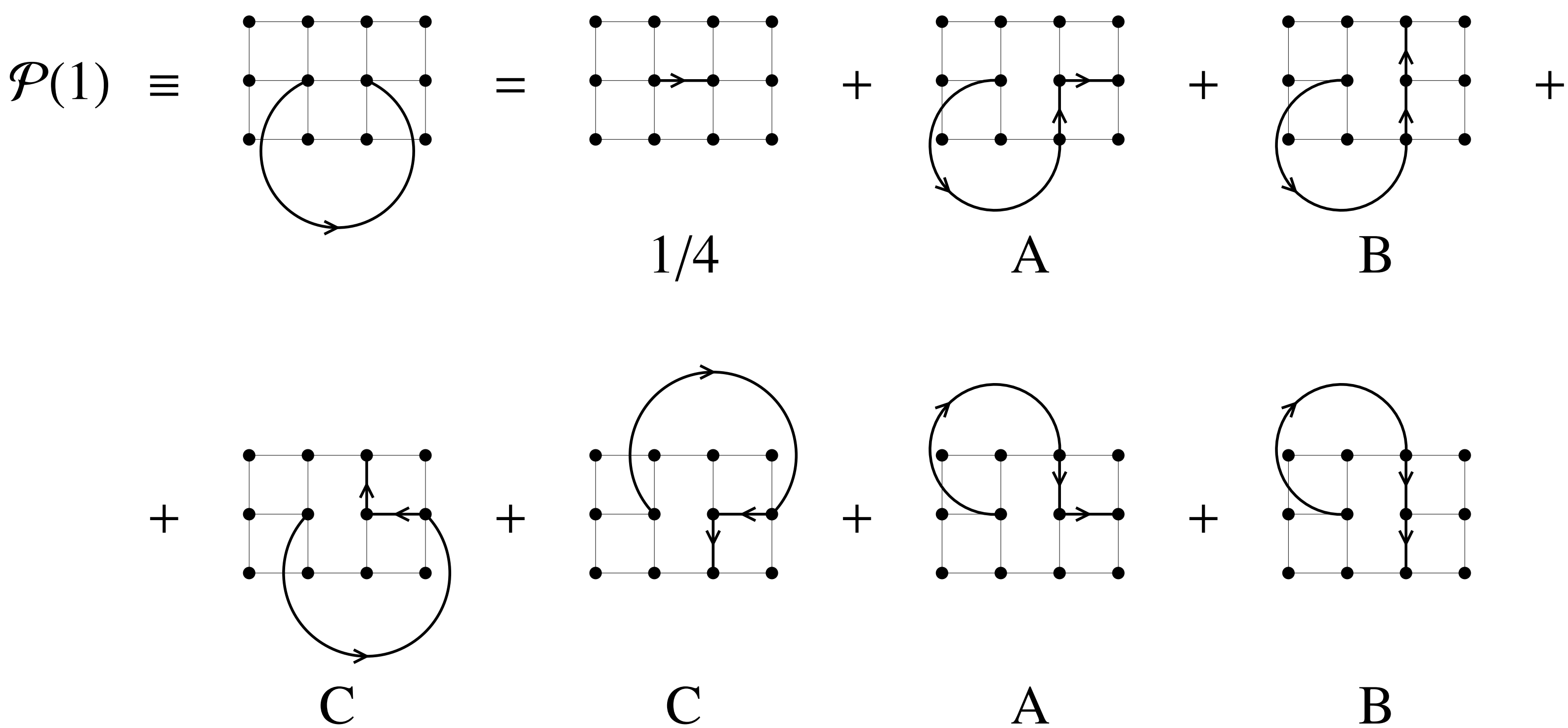


(a) (b) (c)

(b) both paths from i_1 and from i_2 go to the root.

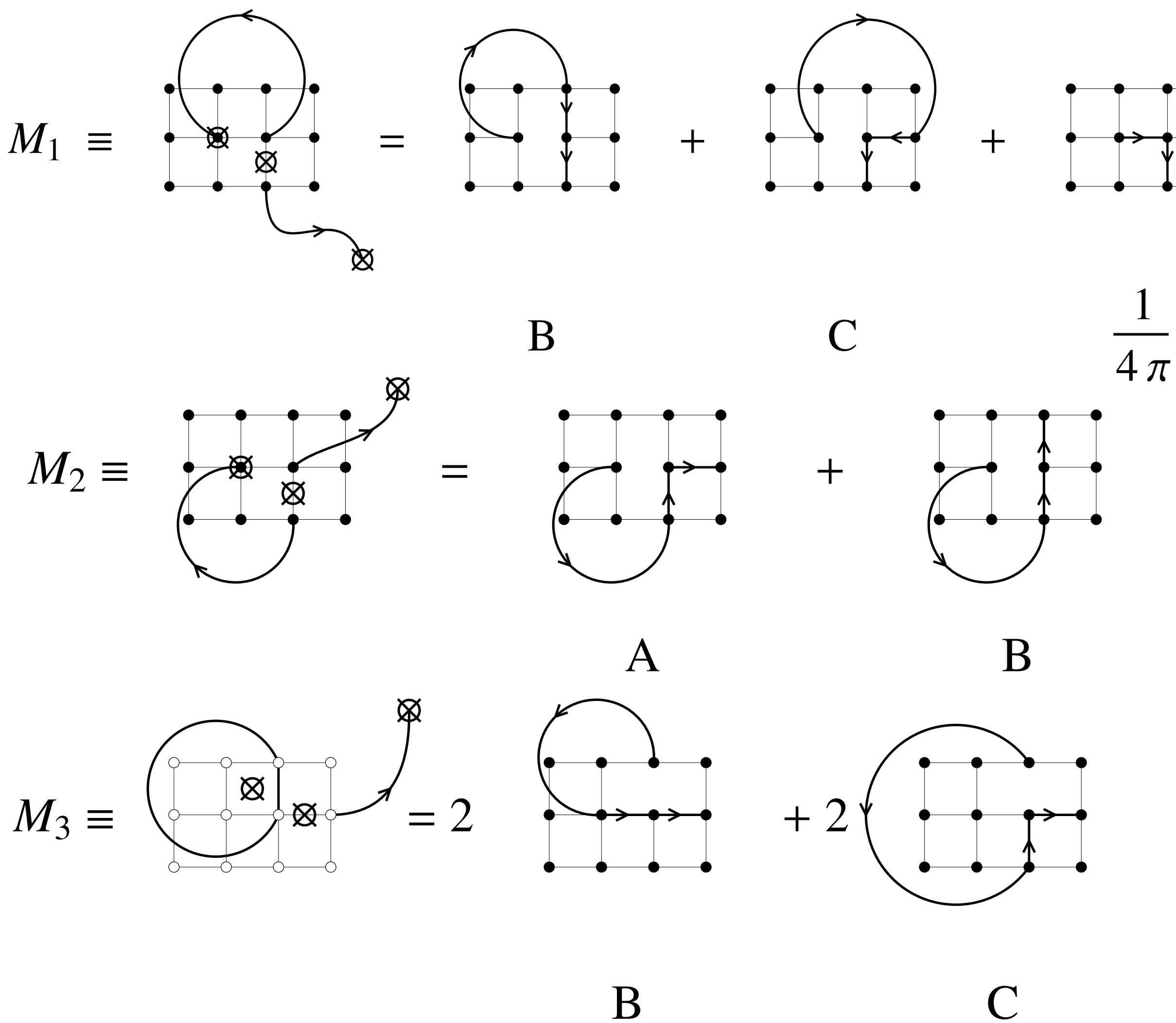
(c) given the spanning tree on the odd-odd sublattice, the single cycle appears on the even-even sublattice with two possible orientations.

All loops contributing to return probability

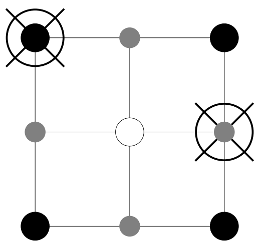


¼ corresponds to the elementary step

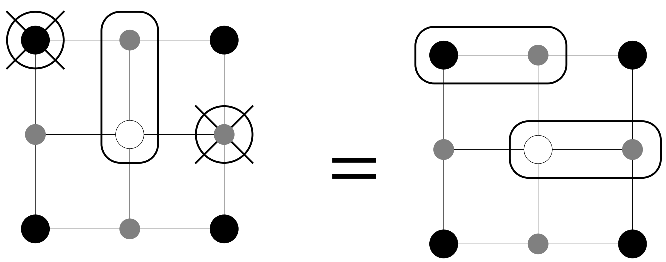
Loops contributing to M_1, M_2, M_3



Monomer-monomer, monomer-dimer, dimer-dimer correlations (Fisher and Stephenson, 1963).

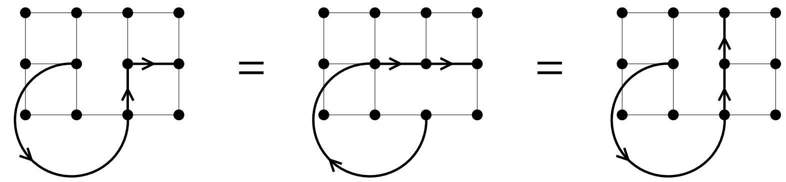


$$P_{mm} = M_1 + M_2 + M_3 = \frac{1}{4\pi} + A + 4B + 3C = \frac{1}{2\pi}$$



$$P_{mdm} = M_2 + \frac{1}{2} M_3 = A + 2B + C = \frac{1}{8} - \frac{1}{4\pi}$$

Solutions for A,B,C

A ≡  ≡ B

$$A = B = \frac{3}{32} - \frac{1}{4\pi}$$

$$C = \frac{1}{2\pi} - \frac{5}{32}$$

$$P_{ret} = \frac{1}{4} + 2(A + B + C)$$

Results (Poghosyan, V.P., Ruelle, 2011)

$$P_{ret} = \frac{5}{16}$$

$$< h >= \frac{25}{8}$$

$$\frac{N_{unicycles}}{N_{ST}} = \frac{1}{8}$$

$$< L_{cycle} >= 8$$

Further development

- $$\frac{1}{8}+\frac{1}{4\pi}+\frac{1}{4\pi^2}-\frac{3}{2\pi^3}+\frac{1}{2\pi^4}$$

- $$\frac{5}{16}$$

- $$\frac{1}{4}-\frac{1}{4\pi}+\frac{1}{2\pi^2}$$

- 1

- $$\frac{5}{16}$$

- $$\frac{1}{8}+\frac{1}{4\pi}+\frac{1}{4\pi^2}-\frac{3}{2\pi^3}+\frac{1}{2\pi^4}$$

Kenyon, Wilson, 2011

Open problems

Coulomb gas prediction (Poghosyan, V.P., 2010)

$$P(r) \simeq \frac{1}{r^{3/4}}$$

Logarithmic conformal field theory prediction (Jeng, Piroux, Ruelle, 2006)

$$P_{22} - P_2^2 \simeq -\frac{P_1^2}{2r^4} \ln^2 r$$