Introduction to the theory of multiplicative chaos

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University Paris-Dauphine

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Plan de l'exposé

1. Motivations
2. Gaussian multiplicative chaos
3. KPZ formula

Plan of the talk

Turbulence

Eddies of a river current

Smoulder and steam of a volcano

Atmospheric turbulence

Wake turbulence

Mathematical approach

The motion of the fluid is ruled by the Navier-Stokes equation:

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u = -\nabla p + \nu \triangle u + f \quad \text{and} \quad \nabla \cdot u = 0.$$

The local dissipation of energy in the set A is defined by:

$$\epsilon(A) = \frac{\nu}{2} \int_A \sum_{i,j} (\partial_i u_j + \partial_j u_i)^2 \, dx.$$

Fully developped turbulence: Kolmogorov 1941

When the velocity of the fluid is "large", the energy dissipation

- is statistically homogeneous and isotrop,
- has linear power-law spectrum (no fluctuations)

$$\mathbb{E}\big[\epsilon(B(0,r))^q\big] \sim C\, r^{\alpha q}.$$

Mathematical legacy of the K41 theory

Kolmogorov, Mandelbrot, Van Ness introduced the Fractional Brownian Motion:

- it is self similar or scale invariant:

$$\forall \lambda > 0, \quad B(\lambda x) \stackrel{law}{=} \lambda^{\alpha} B(x)$$

- linear power law spectrum

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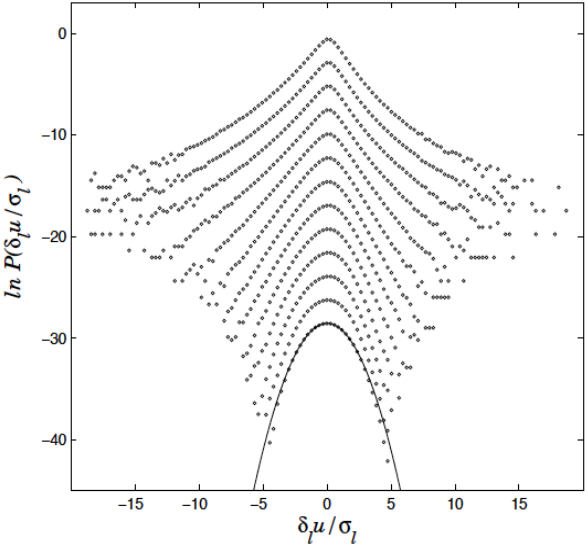


Figure: Probability density function of longitudinal velocity increments $\delta_l u(x) = \langle u(x+le) - u(x), e\rangle$ at different scales l (e is any unit vector)

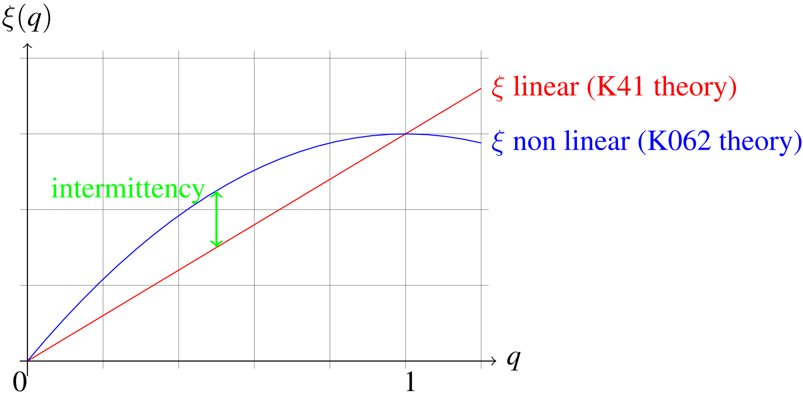
KO62

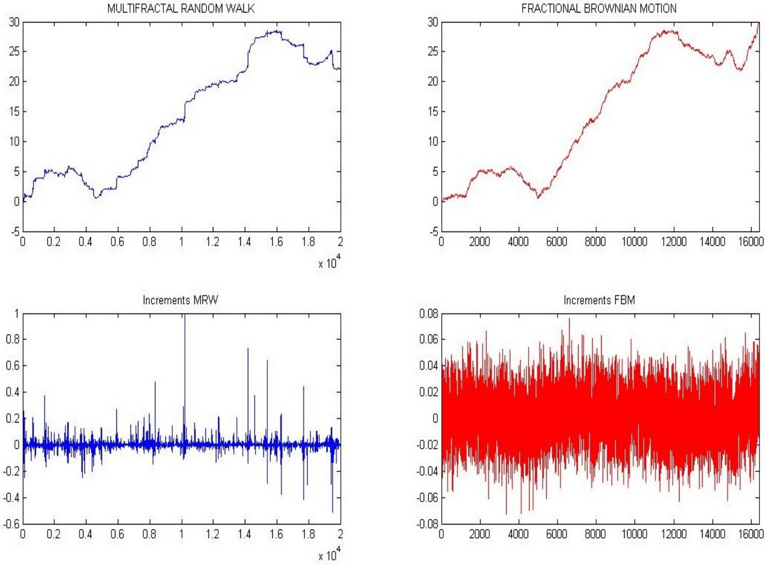
Fully developped turbulence: Kolmogorov-Obukhov 1962

When the velocity of the fluid is "large", the energy dissipation

- is statistically homogeneous and isotrop,
- has non linear power-law spectrum (multifractality)

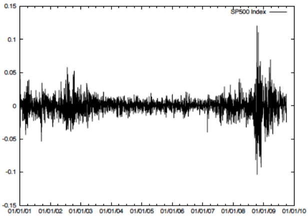
$$\mathbb{E}\Big[\epsilon(B(0,r))^q\Big] \sim C r^{\xi(q)} \quad \text{as } r \to 0.$$



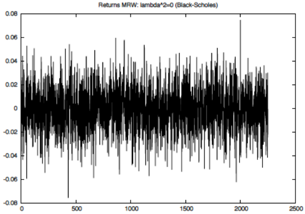


Comparison fractional/multifractal Brownian motion

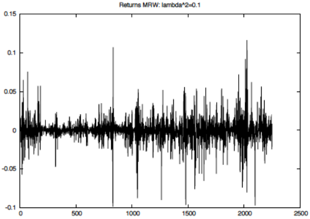
Intermittency in finance



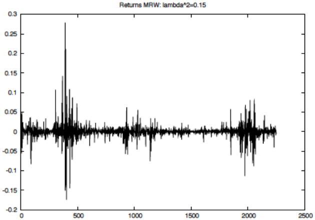
SP500 Returns: 2001-2009



Returns with Black-Scholes



Returns with Multifractal BM



Returns with Multifractal BM

Multifractality

A few names: Frisch, Kahane, Kolmogorov, Mandelbrot,...

Main features:

- intermittency,
- long-range dependence,
- fat tail distribution, pdf of velocity increments

Important subclass

A process is said stochastically scale invariant if:

$$X(\lambda x) \stackrel{law}{=} \lambda^{\alpha} e^{\Omega_\lambda} X(x) \quad \forall\, \lambda \leq 1 \text{ and } x \in B(0,T).$$

where Ω_λ is an infinitely divisible random variable independent of the process X.

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1. Motivations
2. Gaussian multiplicative chaos
3. KPZ formula

Objective

Find a stationary random measure M on $\mathbb{R}^d$ that possesses a nonlinear power law spectrum.

- We look for M in the form

$$M(A) = \int_A e^{X(x) - \frac{1}{2}\mathbb{E}[X(x)^2]}\,dx$$

where X is a centered stationary Gaussian process.

- If the covariance kernel K is continuous at 0 then

$$\mathbb{E}[M(B_r)^q] \simeq |B_r|^q \mathbb{E}[(e^{X(0) - \frac{1}{2}\mathbb{E}[X(0)^2]})^q] = C\, r^{dq}.$$

$\Rightarrow$ linear power law spectrum.

- The kernel K has to be divergent at 0.

$\Rightarrow$ Give sense to the exponential of a random distribution!

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$\Rightarrow$ linear power law spectrum.

- The kernel K has to be divergent at 0.
  $\Rightarrow$ Give sense to the exponential of a random distribution!

Assume K is of σ-positive type (a sum of continuous covariance kernels)

$$K(x,y) = \mathbb{E}[X(x)X(y)] = \sum_n p_n(x,y)$$

1. Let $(X_n)_n$ be a sequence of independent centered Gaussian processes with covariance kernel

$$\mathbb{E}[X_n(x)X_n(y)] = p_n(x,y).$$

2. Define the truncated measure

$$M_n(dx) = \int_{\cdot} \exp\Big(\sum_{k=1}^n X_k(x) - \frac{1}{2}\sum_{k=1}^n \mathbb{E}[X_k^2(x)]\Big)\, dx$$

3. For each set $A \subset \mathbb{R}^d$, the sequence $(M_n(A))_n$ is a positive martingale. Thus it converges towards a limit $M(A)$, called Gaussian multiplicative chaos associated to the kernel K.

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Case of interest

Assume that the covariance kernel K is given by

$$K(x,y) = \mathbb{E}[X(x)X(y)] = \gamma^2 \ln_+ \frac{T}{|x-y|} + g(x,y)$$

where g is bounded and continuous.

Kahane (1985)

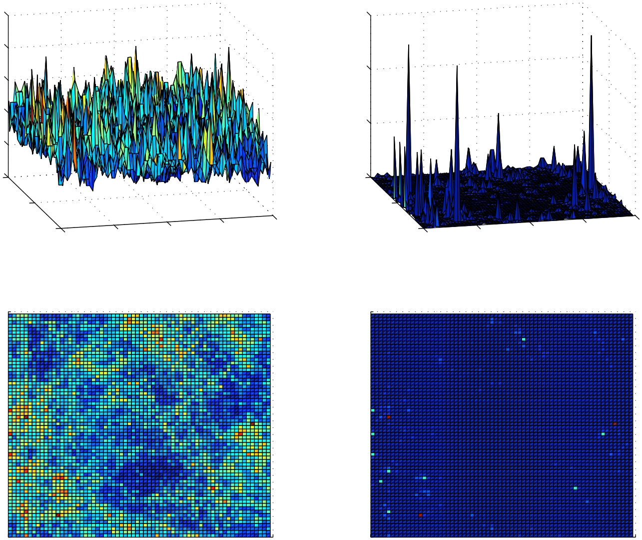
The Gaussian multiplicative chaos M associated to K is different from 0 if and only if

$$\gamma^2 < 2d.$$

Kahane (1985)

For $\gamma^2 < 2d$, the multiplicative chaos M "lives" almost surely on a set with Hausdorff dimension $d - \frac{\gamma^2}{2}$.

2D-density profile: weak/strong intermittence



Example 1: Stoch. Scale Invariance

In dimension $d = 1, 2$ the kernel below is of σ-positive type

$$x \in \mathbb{R}^d \mapsto K(x) = \gamma^2 \ln_+ \left(\frac{T}{|x|}\right)$$

Theorem

The associated multiplicative chaos is stochastically scale invariant: $\forall \lambda < 1$

$$\big(M(\lambda A)\big)_{A \subset B(0,T)} \overset{law}{=} \lambda^d e^{\Omega_\lambda - \frac{1}{2}\mathbb{E}[\Omega_\lambda^2]} \big(M(A)\big)_{A \subset B(0,T)}$$

where Ω_λ *is a centered Gaussian variable with variance* $\gamma^2 \ln \frac{1}{\lambda}$ *independent of* $\big(M(A)\big)_{A \subset B(0,T)}$.

Rhodes, Vargas 2009

There exist stochastically scale invariant multiplicative chaos in dimension $d \geq 3$.

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Hence

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We deduce for $A \subset B(0, T)$

$$\begin{aligned} M(\lambda A) &= \int_{\lambda A} e^{X(x) - \frac{1}{2}\mathbb{E}[X(x)^2]}\, dx \\ &= \lambda^d \int_A e^{X(\lambda y) - \frac{1}{2}\mathbb{E}[X(\lambda y)^2]}\, dy \\ &\overset{law}{=} \lambda^d e^{\Omega_\lambda - \frac{1}{2}\mathbb{E}[\Omega_\lambda^2]} \int_A e^{X(y) - \frac{1}{2}\mathbb{E}[X(y)^2]}\, dy \end{aligned}$$

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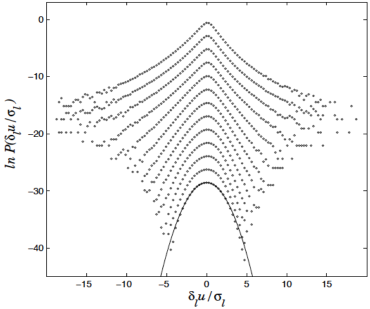
Example 2: Turbulence



Castaing-Gagne-Hopfinger's equation 1990

The local energy dissipation M satisfies the cascading equation:

$$\forall \epsilon \in]0,1], \quad M(dx) \overset{law}{=} e^{X^\epsilon(x)}\ \epsilon M(\frac{dx}{\epsilon})$$

where X^ϵ is a Gaussian process independent of M.

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Theorem: Allez, Rhodes, Vargas (2011)

All the solutions of the above equation are Gaussian multiplicative chaos with kernel of the type:

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Example 3: Quantum measure

Consider the measure

$$M(A) = \int_A e^{X(x) - \frac{1}{2}\mathbb{E}[X(x)^2]} \, dx$$

where X is a Gaussian Free Field in a domain $D \subset \mathbb{R}^2$, that is a Gaussian distribution with covariance kernel

$$K(x,y) = \gamma^2 G(x,y)$$

G= Green function of the Laplacian $\triangle$ on D,

that is

$$\triangle G(x,\cdot) = -2\pi\delta_x.$$

Example 3: Liouville quantum gravity

The function G is of σ positive type:

$$G(x,y) = 2\pi \int_0^{\infty} p_D(t,x,y)\,dt = \sum_{n\in\mathbb{Z}} p_n(x,y)$$

with $p_n(x,y) = 2\pi \int_{2^n}^{2^{n+1}} p_D(t,x,y)\,dt$, and p_D are the transition densities of the symmetric semigroup associated to $\triangle$ with 0 Dirichlet boundary condition.

Theorem

For some continuous bounded function $\bar{G}$:

$$K(x,y) = \gamma^2 \ln \frac{1}{|x-y|} + \bar{G}(x,y).$$

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3. KPZ formula

How to measure dimensions of sets?

Given a Radon measure μ on $\mathbb{R}^d$, define the s-dimensional μ-Hausdorff measure:

$$H^s_\mu(A)=\lim_{\delta\to 0}\inf\Big\{\sum_k \mu(B_k)^{s/d};\ A\subset\bigcup_k B_k,\ \mathrm{diam}(B_k)\leq\delta\Big\}.$$

μ Hausdorff dimension

It is defined as the value

$$\dim_\mu(A)=\inf\{s\geq 0;\ H^s_\mu(A)=0\}.$$

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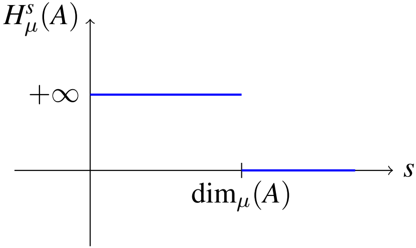

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KPZ formula

Consider a multiplicative chaos

$$M(\cdot) = \int_{\cdot} e^{X(x) - \frac{1}{2}\mathbb{E}[X^2(x)]}\, dx$$

associated to the kernel ($\gamma^2 < 2d$)

$$K(x,y) = \gamma^2 \ln_+ \left(\frac{T}{|x-y|}\right) + g(x,y)$$

Problem

For a given compact set $A \subset \mathbb{R}^d$, find a relation between

$$\dim_{Leb}(A) \quad \text{and} \quad \dim_M(A).$$

KPZ formula

Almost surely, we have the relation

$$\dim_{Leb}(A) = \xi\left(\frac{\dim_M(A)}{d}\right)$$

where

$$\xi(q) = (d + \frac{\gamma^2}{2})q - \frac{\gamma^2}{2}q^2$$

is the power-law spectrum of the chaos measure M, ie:

$$\mathbb{E}\left[M\big(B(0,r)\big)^q\right] \simeq C r^{\xi(q)} \quad \text{as } r \to 0.$$

- I.Benjamini, O.Schramm: KPZ in one dimensional geometry of multiplicative cascades (2008)
- B. Duplantier, S. Sheffield: Liouville Quantum Gravity and KPZ (2008)
- R.Rhodes, V.Vargas: KPZ formula for log-infinitely divisible multifractal random measures (2008)

Remark 1: Quantum measure

When $d = 2$ and M is the Quantum measure (associated to the GFF) we recover the original KPZ relation

$$\dim_{Leb}(A) = \big(1 + \frac{\gamma^2}{4}\big)\dim_M(A) - \frac{\gamma^2}{8}\dim_M(A)^2.$$

Remark 2: Rhodes, Vargas (2008)

More generally, this remains true for any Multifractal Random Measure M regardless of the dimension:

$$\dim_{Leb}(A) = \xi\Big(\frac{\dim_M(A)}{d}\Big)$$

where

$$\xi(q) = dq - \psi(q)$$

is the power law spectrum of M and ψ can be the Laplace exponent of any infinitely divisible random variable.

Heuristic

Remind of the s-dimensional Hausdorff measure

$$H_M^s(A) = \lim_{\delta \to 0} \inf \Big\{ \sum_n M(B_{x_n,r_n})^{\frac{s}{d}}; \quad A \subset \bigcup_n B_{x_n,r_n},\ r_n \leq \delta \Big\}.$$

Take the expectation:

$$\mathbb{E}[H_M^s(A)] = \lim_{\delta \to 0} \inf \Big\{ \sum_n \mathbb{E}[M(B_{x_n,r_n})^{\frac{s}{d}}]; \quad A \subset \bigcup_n B_{x_n,r_n},\ r_n \leq \delta \Big\}$$

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Proof of the KPZ formula

Choose the dimension $d = 1$ and consider the measure

$$M(\cdot) = \int_{\cdot} e^{X_x - \frac{1}{2}\mathbb{E}[X_x^2]}\, dx$$

associated to the kernel

$$K(x) = \gamma^2 \ln_+ \frac{T}{|x|}.$$

The difficult part is to prove

$$\xi(\dim_M(A)) \geq \dim_{Leb}(A).$$

Basic tool: Frostman's lemma:
Each time you have a $q > 0$ and a probability measure ν supported by A such that

$$\int\int \frac{1}{|y-x|^{\xi(q)}}\nu(dx)\nu(dy) < +\infty,$$

find a probability measure $\bar{\nu}$ supported by A such that almost surely:

$$\int\int \frac{1}{M([x,y])^q}\bar{\nu}(dx)\bar{\nu}(dy) < +\infty.$$

Choose

$$\bar{\nu}(\cdot) = \int_{\cdot} e^{qX_x - \frac{q^2}{2}\mathbb{E}[X_x^2]}\,\nu(dx).$$

Basic tool: Frostman's lemma:
Each time you have a $q > 0$ and a probability measure ν supported by A such that

$$\int\int \frac{1}{|y-x|^{\xi(q)}}\nu(dx)\nu(dy) < +\infty,$$

find a probability measure $\bar{\nu}$ supported by A such that almost surely:

$$\int\int \frac{1}{M([x,y])^q}\bar{\nu}(dx)\bar{\nu}(dy) < +\infty.$$

Choose

$$\bar{\nu}(\cdot) = \int_{\cdot} e^{qX_x - \frac{q^2}{2}\mathbb{E}[X_x^2]}\,\nu(dx).$$

$$\mathbb{E}\Big[\int\int \frac{1}{M([x,y])^q}\bar{\nu}(dx)\bar{\nu}(dy)\Big]$$

$$= \int\int \mathbb{E}\Big[\frac{e^{qX_x+qX_y-q^2\mathbb{E}[X_0^2]}}{M([x,y])^q}\Big]\nu(dx)\nu(dy)$$

$$= \int\int \mathbb{E}\Big[\frac{e^{qX_0+qX_{y-x}-q^2\mathbb{E}[X_0^2]}}{M([0,y-x])^q}\Big]\nu(dx)\nu(dy)$$

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We use the scale relation $X_{|y-x|u} = \Omega_{|y-x|} + X_u$ in law, where $\Omega_{|y-x|}, X_u$ are independent and $\Omega_{|y-x|}$ is centered Gaussian with variance $\gamma^2 \ln \frac{1}{|y-x|}$.

$$\mathbb{E}\Big[\int\int \frac{1}{M([x,y])^q}\bar{\nu}(dx)\bar{\nu}(dy)\Big]$$
$$= \int\int \mathbb{E}\Big[\frac{e^{qX_0+qX_{y-x}-q^2\mathbb{E}[X_0^2]}}{M([0,y-x])^q}\Big]\nu(dx)\nu(dy)$$
$$= \int\int \mathbb{E}\Big[\frac{e^{2q\Omega_{|y-x|}-q^2\mathbb{E}[\Omega_{|y-x|}^2]+qX_0+qX_1-q^2\mathbb{E}[X_0^2]}}{|y-x|^q e^{q\Omega_{|y-x|}-\frac{q}{2}\mathbb{E}[\Omega_{|y-x|}^2]}M([0,1])^q}\Big]\nu(dx)\nu(dy)$$

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$$\mathbb{E}\Big[\int\int \frac{1}{M([x,y])^q}\bar{\nu}(dx)\bar{\nu}(dy)\Big]$$

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$$= \int\int \mathbb{E}\Big[\frac{e^{q\Omega_{|y-x|}-(q^2-\frac{q}{2})E[\Omega^2_{|y-x|}]}}{|y-x|^q}\Big]\mathbb{E}\Big[\frac{e^{qX_0+\bar{q}X_1-q^2\mathbb{E}[X_0^2]}}{M([0,1])^q}\Big]\nu(dx)\nu(dy)$$

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$$= \int\int \frac{1}{|y-x|^{\xi(q)}}\mathbb{E}\Big[\frac{e^{qX_0+\bar{q}X_1-q^2\mathbb{E}[X_0^2]}}{M([0,1])^q}\Big]\nu(dx)\nu(dy)$$

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$$\mathbb{E}\Big[\int\int \frac{1}{M([x,y])^q}\bar{\nu}(dx)\bar{\nu}(dy)\Big]$$
$$= \int\int \mathbb{E}\Big[\frac{e^{qX_0+qX_{y-x}-q^2\mathbb{E}[X_0^2]}}{M([0,y-x])^q}\Big]\nu(dx)\nu(dy)$$
$$= C\int\int \frac{1}{|y-x|^{\xi(q)}}\nu(dx)\nu(dy)$$

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$$\mathbb{E}\Big[\int\int \frac{1}{M([x,y])^q}\bar{\nu}(dx)\bar{\nu}(dy)\Big]$$
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$$= C\int\int \frac{1}{|y-x|^{\xi(q)}}\nu(dx)\nu(dy) < +\infty$$

We use the scale relation $X_{|y-x|u} = \Omega_{|y-x|} + X_u$ in law, where $\Omega_{|y-x|}, X_u$ are independent and $\Omega_{|y-x|}$ is centered Gaussian with variance $\gamma^2 \ln \frac{1}{|y-x|}$.

Thank You