Extreme statistics of vicious walkers: from random matrices to Yang-Mills theory

G. Schehr

Laboratoire de Physique Théorique
CNRS-Université Paris Sud-XI, Orsay

Collaborators:

- A. Comtet (LPTMS, Orsay)
- P. J. Forrester (Dept. of Math., Melbourne)
- S. N. Majumdar (LPTMS, Orsay)
- J. Rambeau (LPT, Orsay)
- J. Randon-Furling (Univ. Paris 1, Paris)

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References:

- G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling, Phys. Rev. Lett. **101**, 150601 (2008)
- J. Rambeau, G. S., Europhys. Lett. **91**, 60006 (2010); Phys. Rev. E **83**, 061146 (2011)
- P. J. Forrester, S. N. Majumdar, G. S., Nucl. Phys. B **844**, 500 (2011)

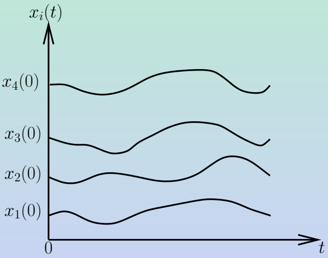
Non-intersecting Brownian motions in 1d

- N Brownian motions in one-dimension

$$\dot{x}_i(t) = \zeta_i(t) \; , \; \langle \zeta_i(t) \zeta_j(t') \rangle = \delta_{i,j} \delta(t - t')$$
$$x_1(0) < x_2(0) < ... < x_N(0)$$

- Non-intersecting condition

$$x_1(t) < x_2(t) < ... < x_N(t) \; , \quad \forall t \geq 0$$



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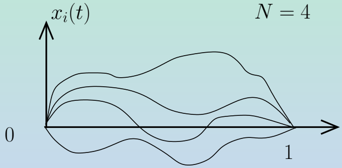

watermelons

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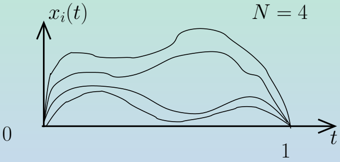


watermelons "with a wall"

Vicious walkers in physics

THE JOURNAL OF CHEMICAL PHYSICS VOLUME 48, NUMBER 5 1 MARCH 1968

Soluble Model for Fibrous Structures with Steric Constraints

P.-G. de Gennes

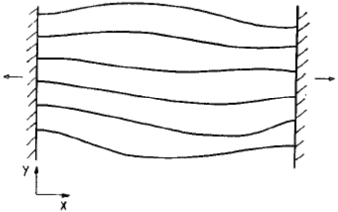
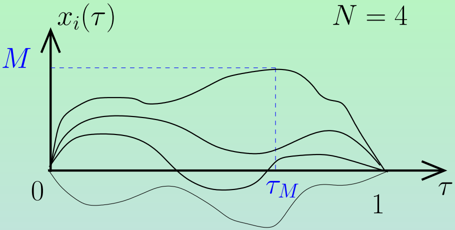


Fig. 1. Model for a two-dimensional fiber structure. The component chains are assumed to be attached to two plates I and F and placed under tension. The chains are bent by thermal fluctuations. Different chains cannot intersect each other.

Vicious walkers in physics

- P. G. de Gennes, *Soluble Models for fibrous structures with steric constraints* (1968)
- M. E. Fisher, *Walks, Walls, Wetting and Melting* (1984)
- B. Duplantier *Statistical Mechanics of Polymer Networks of Any Topology* (1989)
- J. W. Essam, A. J. Guthmann, *Vicious walkers and directed polymer networks in general dimensions* (1995)
- H. Spohn, M. Praehofer, P. L. Ferrari et al. *Stochastic growth models* (2006)
- ...

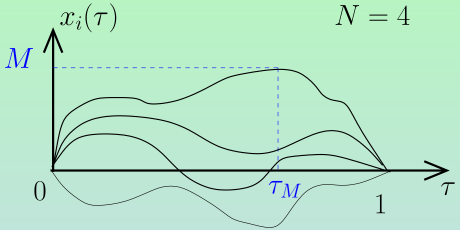
Extreme statistics of vicious walkers



$$x_1(t) < x_2(t) < ... < x_N(t)$$
$$M = \max_{\tau}[x_N(\tau), 0 \leq \tau \leq 1]$$
$$x_N(\tau_M) = M$$

$P_N(M, \tau_M) \equiv$ joint probability distribution function of M, τ_M

Extreme statistics of vicious walkers



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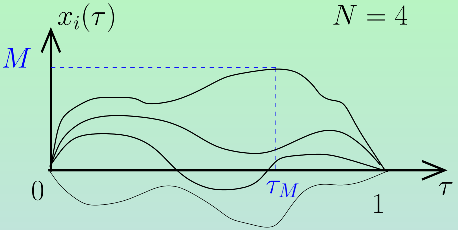
Extreme statistics of vicious walkers



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Q1 : Can one compute $P_N(M, \tau_M)$?

Q2 : Asymptotics of $P_N(M, \tau_M)$ for large N ?

Outline

1. Vicious walkers and random matrices
2. Connection with stochastic growth models
3. Exact computation using path integral approach
4. Watermelons with a wall and connection to Yang-Mills theory
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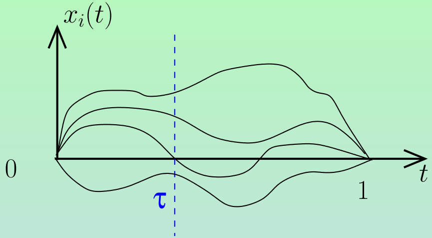
Non intersecting Brownian motions and RMT



- Joint probability of $x_1(\tau), x_2(\tau), \cdots, x_N(\tau)$ at fixed time τ

$$P_{\text{joint}}(x_1, x_2, \cdots, x_N, \tau) \propto \sigma(\tau)^{-N^2} \prod_{i<j=1}^{N} (x_i - x_j)^2 e^{-\frac{1}{\sigma^2(\tau)} \sum_{i=1}^N x_i^2}$$

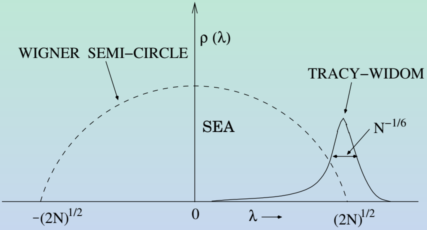
$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of the Gaussian Unitary Ensemble (GUE, $\beta = 2$)

Non intersecting Brownian motions and RMT

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of Gaussian Unitary Ensemble (GUE, $\beta = 2$)
- Mean density $\rho(\lambda)$ of eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$ for GUE

$$\rho(\lambda) = \frac{1}{N} \sum_{\alpha=1}^{N} \langle \delta(\lambda - \lambda_\alpha) \rangle$$

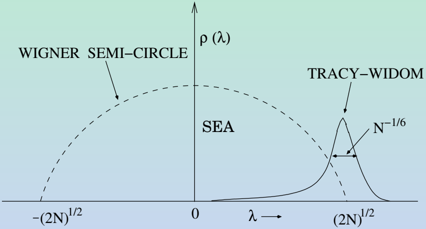


Non intersecting Brownian motions and RMT

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of Gaussian Unitary Ensemble (GUE, $\beta = 2$)
- Largest eigenvalue of random matrices from GUE

$$
\begin{aligned}
\lambda_{\max} &= \max_{1 \leq i \leq N} \lambda_i \\
&= \sqrt{2N} + \frac{N^{-\frac{1}{6}}}{\sqrt{2}} \chi_2 \\
\Pr[\chi_2 \leq \xi] &= \mathcal{F}_2(\xi)
\end{aligned}
$$

Tracy – Widom distribution



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Tracy – Widom distribution

$$\mathcal{F}_2(\xi) = \exp\left[-\int_\xi^\infty (s-\xi) q^2(s)\,\mathrm{d}s\right]$$

where $q(s)$ satisfies Painlevé II

$$\begin{aligned} q''(s) &= s\,q(s) + q^3(s) \\ q(s) &\sim \mathrm{Ai}(s)\ , \ s \to \infty \end{aligned}$$

C. Tracy, H. Widom '94

Watermelons in the limit of large N

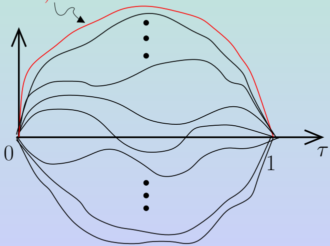
- Consequences for watermelons without wall for large N

$$\frac{x_N(\tau)}{\sqrt{2\tau(1-\tau)}} \sim \sqrt{2N} + \frac{N^{-1/6}}{\sqrt{2}}\chi_2$$

$$\mathrm{Proba}[\chi_2 \leq \xi] = \mathcal{F}_2(\xi), \text{ Tracy-Widom distribution for } \beta = 2$$

- When $N \to \infty$, $x_N(\tau)$ reaches a circular shape

$x_N(\tau) \sim 2\sqrt{N}\sqrt{\tau(1-\tau)}$



Fluctuations

$$x_N(\tau = 1/2) - \sqrt{N} \sim N^{-\frac{1}{6}}$$

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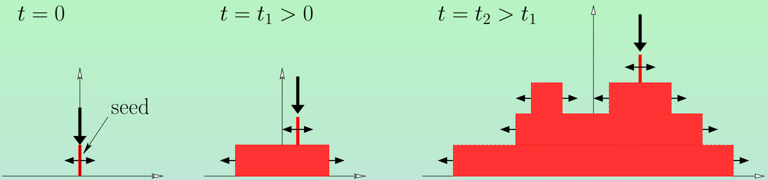
Curved growing interface : the PNG droplet

- Polynuclear Growth Model : Kardar Parisi Zhang universality class



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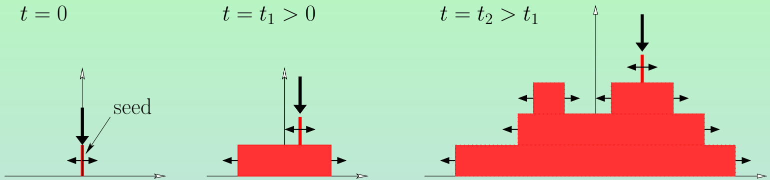
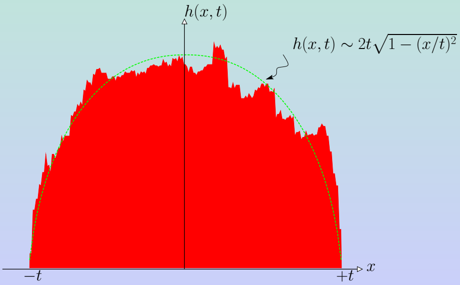


- At large time t the profile becomes droplet-like



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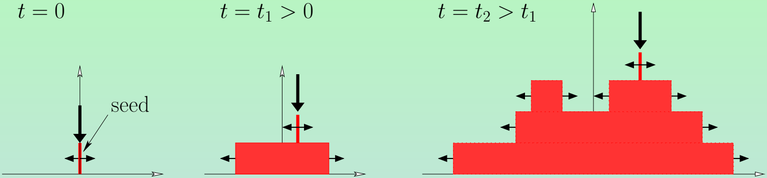

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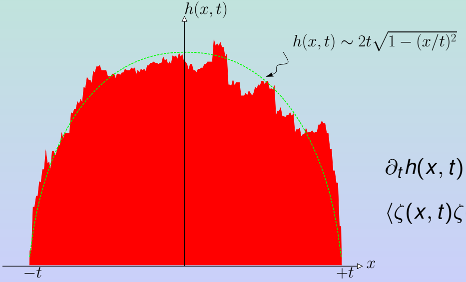


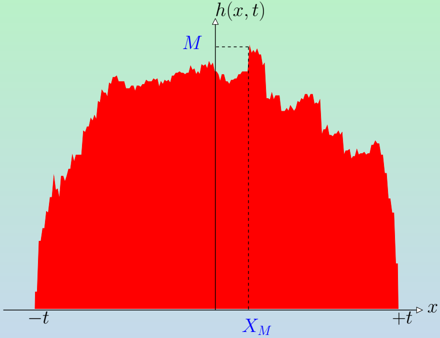
Fluctuations : KPZ equation

$$\partial_t h(x,t) = \nu \nabla^2 h(x,t) + \frac{\lambda}{2}(\nabla h(x,t))^2 + \zeta(x,t)$$

$$\langle \zeta(x,t)\zeta(x',t')\rangle = D\delta(x-x')\delta(t-t')$$

Curved growing interface : the PNG droplet

- Fluctuations : focus on extreme statistics



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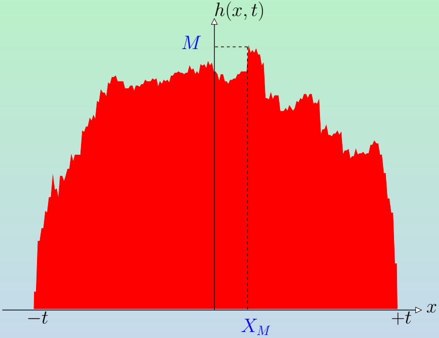

KPZ scaling

$$M - 2t \sim t^{1/3}$$
$$X_M \sim t^{2/3}$$

Curved growing interface : the PNG droplet

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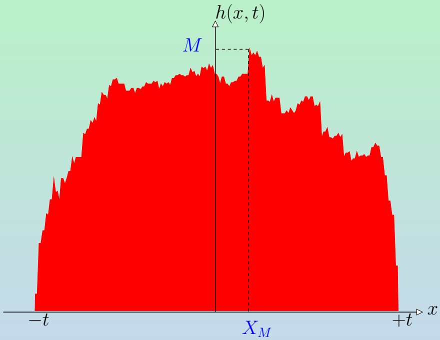


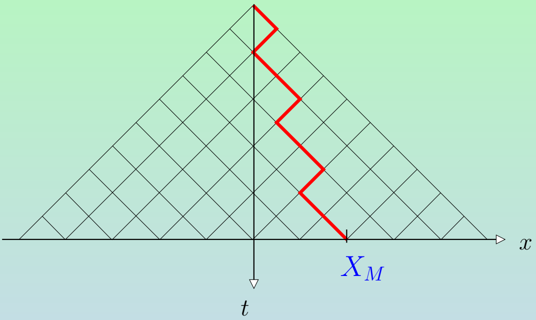
KPZ scaling

$$M - 2t \sim t^{1/3}$$
$$X_M \sim t^{2/3}$$

What is the joint distribution of M, X_M ?

Connection with the Directed Polymer (DPRM)

- DP in random media with one free end ("point to line")

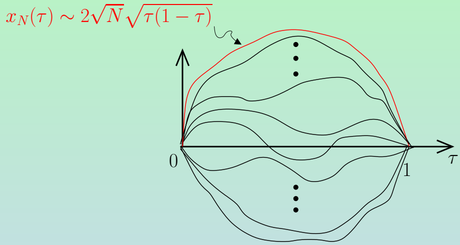


$$E(\mathcal{C}) = \sum_{\langle i,j \rangle \in \mathcal{C}} \epsilon_{ij}$$

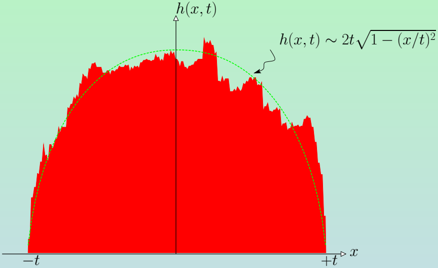
- $M \equiv -$Energy of the optimal polymer
- $X_M \equiv$ Transverse coordinate of the optimal polymer

Vicious walkers and PNG droplet

watermelons



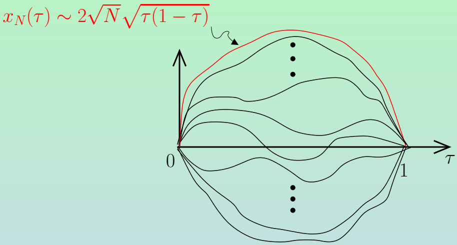
PNG droplet



$$x_N \iff h$$
$$\tau \iff x$$
$$N \iff t$$

Vicious walkers and PNG droplet

watermelons



PNG droplet

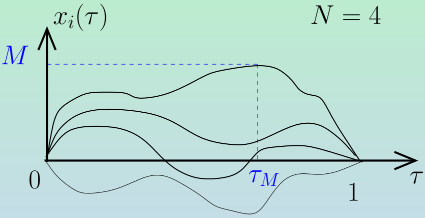
$$\frac{h(u t^{\frac{2}{3}}, t) - 2t}{t^{\frac{1}{3}}} \stackrel{d}{=} \frac{x_N(\frac{1}{2} + \frac{u}{2} N^{-\frac{1}{3}}) - \sqrt{N}}{N^{-\frac{1}{6}}} \stackrel{d}{=} \mathcal{A}_2(u) - u^2$$

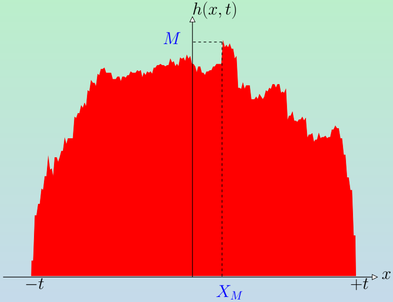
Prähofer & Spohn '00

$\mathcal{A}_2(u) \equiv$ Airy$_2$ process

Vicious walkers and PNG droplet

- Use this correspondence to study extreme statistics of PNG





HERE: exact computation of the distribution $P_N(M, \tau_M)$ for N vicious walkers

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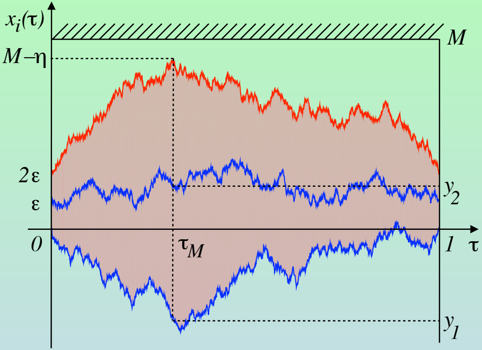
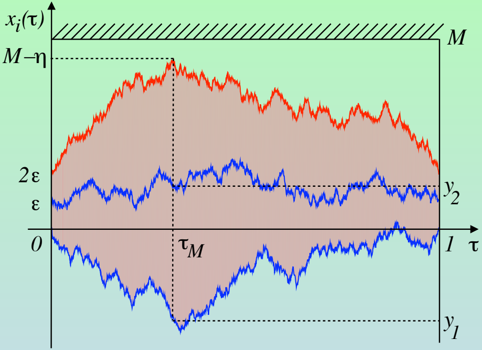
Path integral approach



- $$P_N(M,\tau_M) = \lim_{\epsilon,\eta\to 0} \frac{1}{Z_N} \int_{-\infty}^{M-\eta} \mathrm{d}\mathbf{y}\, p_{<M}(\epsilon,1|\mathbf{y},\tau_M)p_{<M}(\mathbf{y},\tau_M|\epsilon,0)\delta(y_N-(M-\eta))$$

Path integral approach



- $P_N(M, \tau_M) = \lim_{\epsilon,\eta \to 0} \frac{1}{Z_N} \int_{-\infty}^{M-\eta} \mathrm{d}\mathbf{y}\, p_{<M}(\epsilon, 1|\mathbf{y}, \tau_M) p_{<M}(\mathbf{y}, \tau_M|\epsilon, 0) \delta(y_N - (M-\eta))$
- $p_{<M}(\cdot, \cdot|\cdot, \cdot)$: computed using a path integral for free fermions

G.S, S.N. Majumdar, A. Comtet, J. Randon-Furling '08

Exact results for N vicious walkers

- Joint distribution of M and τ_M J. Rambeau, G.S, EPL '10, PRE '11

$$P_N(M, \tau_M) = B_N \, [\det \mathbf{D}] \, {}^t\mathbf{U}(\tau_M) \, \mathbf{D}^{-1} \, \mathbf{U}(1 - \tau_M)$$

$$\mathbf{D}_{i,j} = (-1)^{i-1} H_{i+j-2}(0) - e^{-2M^2} H_{i+j-2}(\sqrt{2}M)$$

$$\mathbf{U}_i(\tau_M) = \tau_M{}^{-\frac{i+1}{2}} H_i \left(M/\sqrt{2\tau_M} \right) \; e^{-\frac{M^2}{2\tau_M}}$$

$H_i(\cdot) \equiv$ Hermite polynomials

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$$H_i(\cdot) \equiv \text{Hermite polynomials}$$

- Marginal distribution of τ_M

$$N = 2 \;:\; P_2(\tau_M) = 4\left(1 - \frac{1+10\tau_M(1-\tau_M)}{(1+4\tau_M(1-\tau_M))^{5/2}}\right)$$

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- Asymptotics for large N ?

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- Asymptotics for large N ? still difficult BUT ...

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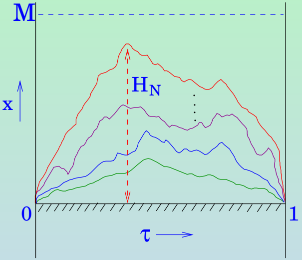
Maximal height of watermelons with a wall

- Cumulative distribution of the maximal height

$$F_N(M) = \Pr\left[x_N(\tau) \leq M \,,\ \forall\, 0 \leq \tau \leq 1\right] = \int_0^1 d\tau_M \int_0^M dx\, P_N(x, \tau_M)$$



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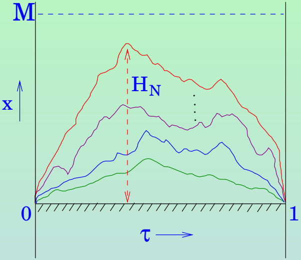

- Path integral for free fermions

G. S, S. N. Majumdar, A. Comtet, J. Randon-Furling '08

$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\leq j<k\leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2M^2}\sum_{i=1}^N n_i^2}$$

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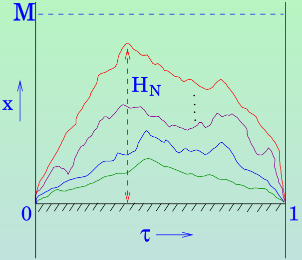


- Path integral for free fermions

$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\leq j<k\leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2M^2}\sum_{i=1}^N n_i^2}$$

What about the asymptotic behavior of $F_N(M)$ for $N \to \infty$?

Partition function of Yang-Mills theory in 2d

- Partition function of Yang-Mills theory on a $2d$ manifold $\mathcal{M}$ with a gauge group G, described by a gauge field $A_\mu(x) \equiv A_\mu^a(x) T^a$

$$\mathcal{Z}_\mathcal{M} = \int [\mathcal{D}A_\mu] e^{-\frac{1}{4\lambda^2} \int \mathrm{Tr}[F^{\mu\nu} F_{\mu\nu}] \sqrt{g} d^2x}$$

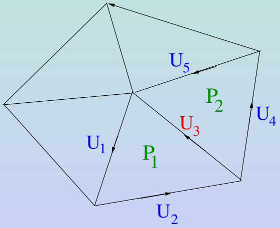
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

Ex: $G \equiv SU(2)$: electro-weak interact°, $G \equiv SU(3)$: chromodynamics

- Regularization on the lattice

$$\mathcal{Z}_\mathcal{M} = \int \prod_L dU_L \prod_{\text{plaquettes}} Z_P[U_P]$$

$$U_P = \prod_{L \in \text{plaquette}} U_L$$



Heat-kernel action

$$\mathcal{Z}_{\mathcal{M}} = \int \prod_{L} dU_L \prod_{\text{plaquettes}} Z_P[U_P]$$

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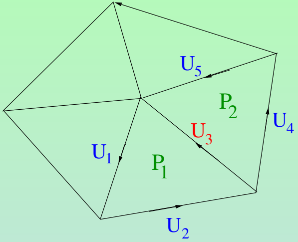

- A common choice : Wilson's action Wilson'74

$$Z_P(U_P) = \exp\left[bN\,\mathrm{Tr}(U_P + U_P^\dagger) \right]$$

Heat-kernel action

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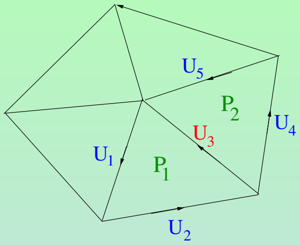

- A common choice : Wilson's action — Wilson'74

$$Z_P(U_P) = \exp\left[bN\,\mathrm{Tr}(U_P + U_P^\dagger)\right]$$

- Alternative choice : invariance under decimation ⇒ Migdal's recursion relation

$$\int dU_3\, Z_{P_1}(U_1 U_2 U_3) Z_{P_2}(U_4 U_5 U_3^\dagger) = Z_{P_1+P_2}(U_1 U_2 U_4 U_5)$$

$$Z_P(U_P) = \sum_R d_R \chi_R(U_P) \exp\left[-\frac{A_P}{2N} C_2(R)\right]$$

Migdal'75, Rusakov'90

Partition function of Yang-Mills theory on the $2d$-sphere

- Partition funct$^{\text{o}}$ on $\mathcal{M}$, of genus g, computed with the heat-kernel action

$$\mathcal{Z}_{\mathcal{M}} = \sum_R d_R^{2-2g} \exp\left[-\frac{A}{2N} C_2(R)\right]$$

Partition function of Yang-Mills theory on the $2d$-sphere

- Partition funct° on the sphere computed with the heat-kernel action

$$\mathcal{Z}_{\mathcal{M}} = \sum_R d_R^2 \exp\left[-\frac{A}{2N} C_2(R)\right]$$

Partition function of Yang-Mills theory on the 2d-sphere

- Partition functº on the sphere computed with the heat-kernel action

$$\mathcal{Z}_{\mathcal{M}} = \sum_{R} d_R^2 \exp\left[-\frac{A}{2N} C_2(R)\right]$$

- Irreducible representations R of G are labelled by the lengths of the Young diagrams:
  - If $G = \mathrm{U}(N)$

$$\mathcal{Z}_{\mathcal{M}} = c_N\, e^{-A\frac{N^2-1}{24}} \sum_{n_1,\ldots,n_N=0}^{\infty} \prod_{i<j} (n_i - n_j)^2 e^{-\frac{A}{2N})\sum_{j=1}^{N} n_j^2}$$

  - If $G = \mathrm{Sp}(2N)$

$$\mathcal{Z}_{\mathcal{M}} = \hat{c}_N\, e^{A(N+\frac{1}{2})\frac{N+1}{12}} \sum_{n_1,\ldots,n_N=0}^{\infty} \left(\prod_{j=1}^{N} n_j^2\right) \prod_{i<j} (n_i^2 - n_j^2)^2 e^{-\frac{A}{4N}\sum_{j=1}^{N} n_j^2}$$

Correspondence between YM_2 on the sphere and watermelons

- Partition function of YM_2 on the sphere with gauge group $\mathrm{Sp}(2N)$

$$\mathcal{Z}_{\mathcal{M}} = \mathcal{Z}(A; \mathrm{Sp}(2N))$$

$$\mathcal{Z}(A; \mathrm{Sp}(2N)) = \hat{c}_N \, e^{A\,(N+\frac{1}{2})\,\frac{N+1}{12}} \sum_{n_1,\ldots,n_N=0}^{\infty} \left(\prod_{j=1}^{N} n_j^2\right) \prod_{i<j} (n_i^2 - n_j^2)^2 \, e^{-\frac{A}{4N}\sum_{j=1}^{N} n_j^2}$$

- Cumulative distribution of the maximal height of watermelons with a wall

$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \left(\prod_{j=1}^{N} n_j^2\right) \prod_{i<j} (n_i^2 - n_j^2)^2 \, e^{-\frac{\pi^2}{2M^2}\sum_{j=1}^{N} n_j^2}$$

$$\propto \mathcal{Z}\left(A = \frac{2\pi^2 N}{M^2}; \mathrm{Sp}(2N)\right)$$

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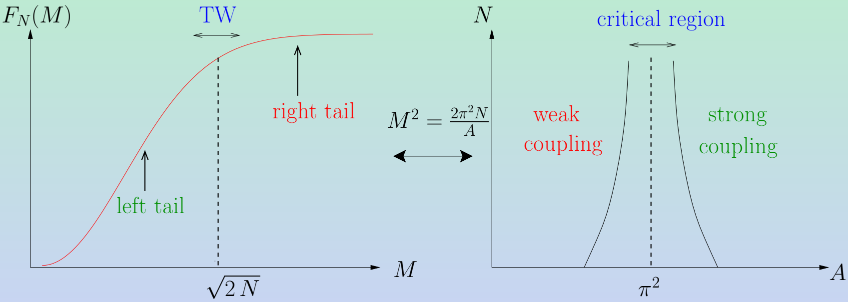
Large N limit of YM_2 and consequences for $F_N(M)$

- Weak-strong coupling transition in YM_2 Durhuus-Olesen '81, Douglas-Kazakov '93



Large N limit of YM_2 and consequences for $F_N(M)$

- In the critical regime, "double-scaling limit", the method of orthogonal polynomials (Gross-Matytsin '94, Crescimanno-Naculich-Schnitzer '96) shows

$$\frac{d^2}{dt^2}\log F_N\left(\sqrt{2N}(1+t/(2^{7/3}N^{2/3}))\right) = -\frac{1}{2}\left(q^2(t)+q'(t)\right)$$
$$q''(t) = 2q^3(t) + t\,q(t)\ ,\ q(t) \sim \mathrm{Ai}(t)\ ,\ t \to \infty$$

i.e.

$$F_N(M) \quad \to \quad \mathcal{F}_1\left(2^{11/6}N^{1/6}\left|M-\sqrt{2N}\right|\right)$$
$$\mathcal{F}_1(t) \quad = \quad \exp\left(-\frac{1}{2}\int_t^\infty ((s-t)\,q^2(s)-q(s))\ ds\right)$$
$$\equiv \quad \text{Tracy-Widom distribution for } \beta = 1$$

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Large N limit of YM_2 and consequences for $F_N(M)$

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$\equiv$ Tracy-Widom distribution for $\beta = 1$

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- Also interesting results for large deviations

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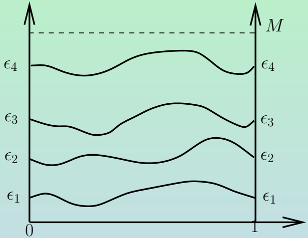
What about other gauge groups ?

- Ratio of reunion probabilities for N vicious walkers on the segment $[0, M]$ with absorbing boundary conditions



$$F_N(M) = \text{Proba}[x_N(\tau) \leq M, \forall \tau \in [0,1]]$$

$$F_N(M) = \frac{R_M(1)}{R_\infty(1)}$$

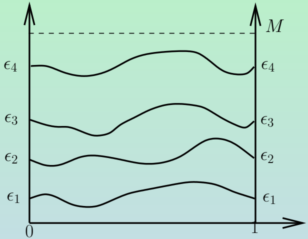
$R_M(1) \equiv$ proba. that N walkers return to their initial positions at $\tau = 1$

Related to YM_2 on the sphere with gauge group $\mathrm{Sp}(2N)$

$$F_N(M) \propto \mathcal{Z}\left(A = \frac{2\pi^2 N}{M^2}; \mathrm{Sp}(2N)\right)$$

What about other gauge groups ?

- Ratio of reunion probabilities for N vicious walkers on the segment $[0, M]$ with periodic boundary conditions



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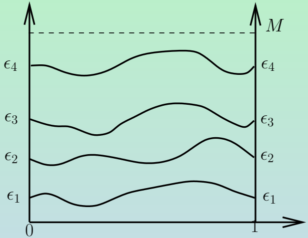
Related to YM_2 on the sphere with gauge group $\mathrm{U(N)}$

$$F_N(M) \propto \mathcal{Z}\left(A = \frac{4\pi^2 N}{M^2}; \mathrm{U}(N)\right)$$

What about other gauge groups ?

- Ratio of reunion probabilities for N vicious walkers on the segment $[0, M]$ with reflecting boundary conditions



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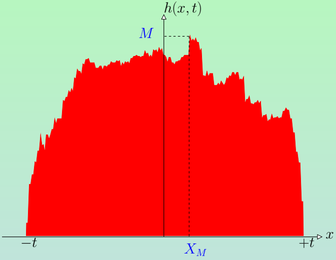
Related to YM_2 on the sphere with gauge group $\mathrm{SO}(2N)$

$$F_N(M) \propto \mathcal{Z}\left(A = \frac{4\pi^2 N}{M^2}; \mathrm{SO}(2N)\right)$$

Consequences for curved stochastic growth

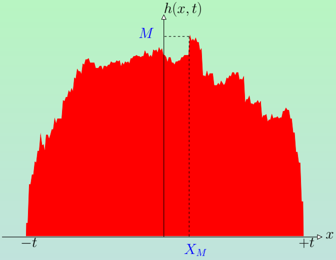


- Distribution of the height field $h(0,t)$ Prähofer & Spohn '00

$$\lim_{t\to\infty} P\left(\frac{h(0,t)-2t}{t^{1/3}} \leq s\right) = \mathcal{F}_2(s)$$

$\mathcal{F}_2(s) \equiv$ Tracy – Widom distribution for $\beta = 2$

Consequences for curved stochastic growth



- Maximum $M \equiv \max_{-t \le x \le t} h(x,t)$ P. Forrester, S. N. Majumdar, G. S. NPB '11

$$\lim_{t \to \infty} P\left(\frac{M - 2t}{t^{1/3}} \le s \right) = \mathcal{F}_1(s)$$

$$\mathcal{F}_1(s) \equiv \text{Tracy} - \text{Widom distribution for } \beta = 1$$

Consequences for curved stochastic growth

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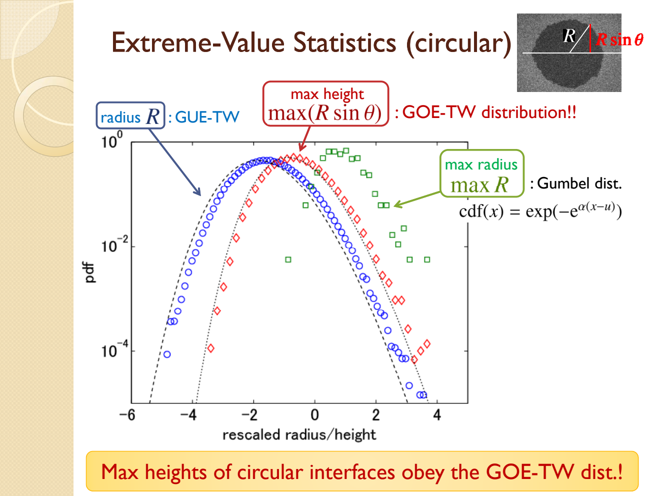
see also

- Krug *et al.* '92, Johansson '03 (indirect proof),
- G. M. Flores, J. Quastel, D. Remenik, arXiv:1106.2716

Experiments on nematic liquid crystals

K. A. Takeuchi, M. Sano, Phys. Rev. Lett. **104**, 230601 (2010)



Courtesy of K. Takeuchi

Outline

1. Vicious walkers and random matrices
2. Connection with stochastic growth models
3. Exact computation using path integral approach
4. Watermelons with a wall and connection to Yang-Mills theory
5. Conclusion

Conclusion

- Exact results for extreme statistics of N vicious walkers
- Relation between vicious walkers and Yang-Mills theory on the sphere
- The maximal height is given, for $N \to \infty$, by the Tracy-Widom distribution $\beta = 1$
- Relation between boundary conditions in vicious walkers problem and gauge group in the YM_2

Conclusion

- Exact results for extreme statistics of N vicious walkers
- Relation between vicious walkers and Yang-Mills theory on the sphere
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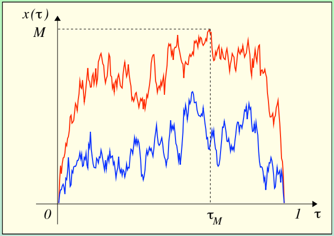
- Relation between boundary conditions in vicious walkers problem and gauge group in the YM_2

Any deep reason behind this ?

Perspectives



$P_N(M, \tau_M) \equiv$ joint distribution of the maximum M and its position τ_M

J. Rambeau, G. S. '11

$$P_N(M,\tau_M) = \frac{A_{N,E}}{M^{N(2N+1)+3}} \sum_{\mathbf{n},n'_N} \Bigg\{ (-1)^{n_N+n'_N}\, n_N^2\, {n'_N}^2 \left(\prod_{i=1}^{N-1} n_i^2\right) e^{-\frac{\pi^2}{2M^2}\sum_{i=1}^{N-1} n_i^2}$$

$$\Delta_N(n_1^2,\ldots,n_{N-1}^2,n_N^2)\,\Delta_N(n_1^2,\ldots,n_{N-1}^2,{n'_N}^2) e^{-\frac{\pi^2}{2M^2}\left[(1-\tau_M){n'_N}^2+\tau_M n_N^2\right]} \Bigg\}$$

What about the large N limit of $P_N(M, \tau_M)$?
see recent results by G. M. Flores, J. Quastel, D. Remenik, arXiv:1106.2716