Conformal blocks and the Calogero-Sutherland model

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arXiv:1109.xxxx

Random Processes, CFT and Integrable Models
Moscow, September 20, 2011

Calogero-Sutherland model and 2d CFT's

- Laughlin wavefunction, c=1 CFT and CS model **[Haldane 90's, ... Haldane, Bernevig 07]**
- Matrix models and collective field representation for CS model **[Jevicki, 92,...]**
- Spinons in su(2)k=1 WZW model **[Bernard, Pasquier, DS, 94]**
- CS and singular vectors of Virasoro algebra **[Awata, Matsuo, Odake, Shiraisi, 95; Arnaudon, Avan, Frappat, Ragoucy, Shiraishi, 06]**
- Quantum hydrodynamics, CS and Benjamin-Ono **[Abanov, Wiegmann, 05]**
- SLE, CFT and CS model **[Cardy, 04; Cardy, Doyon, 07; Dubedat, 06]**
- FQHE with pairing properties, CFT's and CS model **[Nayak, Wilczek, 96; Haldane, Bernevig 07; Estienne, Bernevig, Santachiara, 10]**
- AGT conjecture and CS model **[Alba, Fateev, Litvinov, Tarnopolsky 10; A. Litvinov's talk; Belavin, Belavin 11, ...]**

...

Conformal blocks of some 2d CFT's

$$\langle \Phi_{12}(z_1)\cdots\Phi_{12}(z_N)\Phi_{21}(w_1)\cdots\Phi_{21}(w_M)\rangle$$

$$= \sum_{\lambda}\langle 0|\Phi_{12}(z_1)\cdots\Phi_{12}(z_N)|\lambda\rangle\langle\lambda|\Phi_{21}(w_1)\cdots\Phi_{21}(w_M)|0\rangle$$

FQHE states with non-abelian statistics

[Estienne, Bernevig, Santachiara, 10]

$$\begin{array}{ccc} g & \to & -g \\ b & \to & ib \end{array}$$

AGT conjecture

(Nekrasov's partition function ~ Liouville conformal blocks)

[Alba, Fateev, Litvinov, Tarnopolsky 10]

A. Litvinov's talk

Integrable structure of the CS model

Summary

- CS Hamiltonian and the degenerate fields in CFT
- duality of the conformal blocks
- Ising CFT and FQHE states
- AFLT Hamiltonians for generic Virasoro models
- WA_{k-1} models

Calogero-Sutherland Hamiltonian

Trigonometric CS model: set of N commuting Hamiltonians for N particles on a circle:

$$H_1^g = \mathcal{P} = \sum_{i=1}^{N} z_i \partial_i$$

$$H_2^g = H^g = \sum_{i=1}^{N} (z_i \partial_i)^2 - g(g-1) \sum_{i \neq j} \frac{z_i z_j}{z_{ij}^2} \qquad z_j = e^{2i\pi x_j / L}$$

$$H_3^g = \sum_{i=1}^{N} (z_i \partial_i)^3 + \frac{3}{2} g(1-g) \sum_{i \neq j} \frac{z_i z_j}{z_{ij}^2} (z_i \partial_i - z_j \partial_j).$$

...

Two different **boundary conditions** for the wave functions:

$$\Psi^+(z) = \Delta^g(z) F^+(z) \quad \text{or} \quad \Psi^-(z) = \Delta^{1-g}(z) F^-(z) \qquad \Delta^\gamma(z) = \prod_{i<j} (z_i - z_j)^\gamma$$

Polynomial eigenfunctions (Jack symmetric polynomials) $\longleftrightarrow$ abelian statistics

$$\Psi_\lambda^+(z) = \Delta^g(z) J_\lambda^{1/g}(z) \qquad \Psi_\lambda^-(z) = \Delta^{1-g}(z) J_\lambda^{1/(1-g)}(z)$$

Jack polynomials: eigenfunctions of the Hamiltonian

$$\mathcal{H}^\alpha = \sum_{i=1}^{N}(z_i\partial_i)^2 + \frac{1}{\alpha}\sum_{i<j}^{N}\frac{z_i+z_j}{z_{ij}}(z_i\partial_i - z_j\partial_j)$$

$$\alpha^{-1} = g \quad \text{or} \quad 1-g$$

$$\mathcal{E}_\lambda^\alpha = \sum_i^N \lambda_i\left[\lambda_i + \frac{1}{\alpha}(N+1-2i)\right]$$

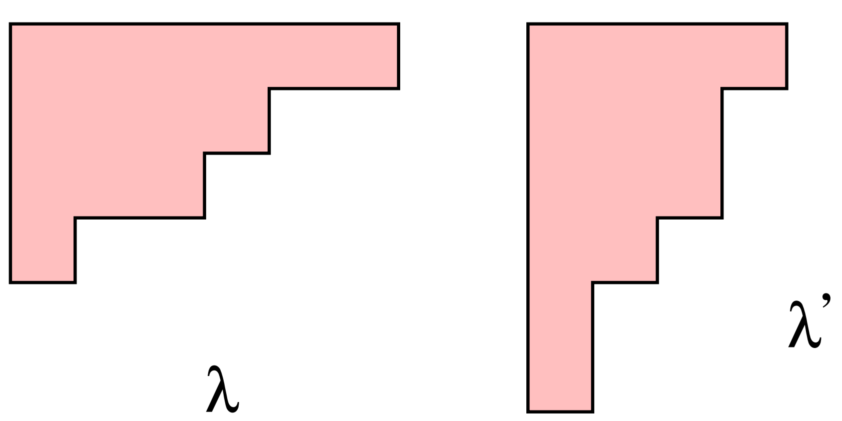
characterized by **partitions** λ with λ_i integers: $\lambda_1 \geq ... \geq \lambda_N \geq 0$

Duality $g \to 1/g$: **[Stanley 89; Macdonald 88; Gaudin 92]**

$$\left[\mathcal{H}^{1/g} + g\,\mathcal{H}^g + C(N,M)\right]\prod_{i=1}^{N}\prod_{i=1}^{M}(1+z_iw_j) = 0$$

$$\prod_{i,j}(1+z_iw_j) = \sum_\lambda J_\lambda^{1/g}(z)J_{\lambda'}^{g}(w)$$

Dual partitions:



Degenerate fields in CFT

Virasoro models with central charge : $c = 1 - 6\frac{(g-1)^2}{g}$

Degenerate field with dimensions : $\Delta_{(r|s)} = \frac{1}{4}\left(\frac{r^2-1}{g} + (s^2-1)g + 2(1-rs)\right)$

Two second-level degenerate fields : $\left(L_{-1}^2 - gL_{-2}\right)\Phi_{(1|2)} = 0\,,$ $\left(L_{-1}^2 - \frac{1}{g}L_{-2}\right)\Phi_{(2|1)} = 0$

When inserted in correlation function, the null-vector conditions translate into differential equations:

$$\mathcal{O}^g(z)\langle\Phi_{(1|2)}(z)\Phi_{\Delta_1}(z_1)\ldots\Phi_{\Delta_N}(z_N)\rangle = 0$$

with

$$\mathcal{O}^g(z) = \frac{\partial^2}{\partial z^2} - g\left(\sum_{j=1}^{N}\frac{\Delta_i}{(z-z_j)^2} + \frac{1}{z-z_j}\frac{\partial}{\partial z_j}\right)$$

Conformal blocks and the duality

Consider the **dressed conformal blocks** :

$$\mathcal{F}^{a,b}_{M,N}(w;z) \equiv \langle \Phi_{21}(w_1)\cdots\Phi_{21}(w_M)\Phi_{12}(z_1)\cdots\Phi_{12}(z_N)\rangle_{a,b} \prod_{1\le i<j}^{M} w_{ij}^{2\widetilde{h}} \prod_{i,j}(w_i - z_j)^{1/2} \prod_{1\le i<j}^{N} z_{ij}^{2h}$$

$$h = \Delta_{12} = \frac{3g}{4} - \frac{1}{2} \,, \qquad \widetilde{h} = \Delta_{21} = \frac{3}{4g} - \frac{1}{2}$$

CS action on the conformal blocks (M=0 case: **[Cardy 04]**)

Duality :

$$\left[h^{\alpha}(z) + g\, h^{\widetilde{\alpha}}(w)\right] \mathcal{F}^{a,b}_{M,N}(w;z) = 0$$

with dual CS Hamiltonians

$$h^{\alpha}(z) \equiv \mathcal{H}^{\alpha}(z) - \mathcal{E}^{\alpha}_0 + \left(\frac{N-2}{\alpha} - 1\right)[\mathcal{P}(z) - \mathcal{P}_0] - \frac{NM(M-2)}{4} \,,$$
$$h^{\widetilde{\alpha}}(w) \equiv \mathcal{H}^{\widetilde{\alpha}}(w) - \mathcal{E}^{\widetilde{\alpha}}_0 + \left(\frac{M-2}{\widetilde{\alpha}} - 1\right)[\mathcal{P}(w) - \mathcal{P}'_0] - \frac{NM(N-2)}{4} \,,$$

and dual coupling constants

$$\alpha^{-1} = 1 - g \,, \qquad \widetilde{\alpha}^{-1} = 1 - g^{-1}$$

non-abelian generalization of the Stanley-Macdonald-Gaudin duality

$$\mathcal{F}^{a,b}_{M,N}(w;z) = \sum_{\lambda} P^{\widetilde{\alpha},a}_{\lambda'}(w)\, P^{\alpha,b}_{\lambda}(z)$$

[Estienne, Bernevig, Santachiara 10]
for Z_k parafermionic CFTs

Ising CFT and the Moore-Read FQHE wave-function

[Moore, Read, 91]

$\Psi = \Phi_{12}$ electron

$\sigma = \Phi_{21}$ quasihole

$g = 4/3$

$\alpha = 1/(1-g) = -3$

The electron eigenfunction is **monovalued**:

Moore-Read wavefunction $\sim$

$$\prod_{i<j} z_{ij}^2 \, \langle \Psi(z_1) \dots \Psi(z_N) \rangle = \prod_{i<j} z_{ij}^2 \, \mathrm{Pf}\left(\frac{1}{z_{ij}} \right) = \prod_{i<j} z_{ij} \, J_{\lambda_0}^{-3}(z)$$

$$\lambda_0 = [N-2, N-2, N-4, N-4, \cdots, 0, 0]$$

with **clustering properties** (it vanishes when a cluster of 3 particles come together):

eigenfunction of a three-body Hamiltonian

$$H = \sum_{i \neq j \neq k} \delta^{(2)}(x_i - x_j) \delta^{(2)}(x_j - x_k)$$

Generic (k, r) clustering properties of Jack polynomials for coupling constant $\alpha = -(k+1)/(r-1)$

[Feigin, Miwa, Jimbo, Mukhin, 02]

- the coefficients of the Jack polynomials diverge except for **admissible partitions**

(k, r, N)-admissible partition λ : $\quad \lambda_i - \lambda_{i+k} \geq r \quad (1 \leq i \leq N-k)$

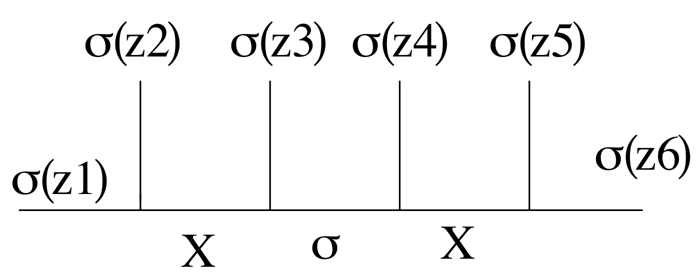
$J_\lambda^{-(k+1)/(r-1)}(z_1, \cdots, z_n)$ vanishes when $z_1 = z_2 = \cdots = z_{k+1}$

Ising CFT and the Moore-Read FQHE wave-function

quasihole wave-function: $\sim \quad \Psi(z)_a \equiv \prod_{i<j} z_{ij}^{3/8} \langle \sigma(z_1) \ldots \sigma(z_M) \rangle_a$

[Nayak, Wilczek, 96]

it is **multivalued**, with the $2^{M/2-1}$ conformal blocks corresponding to the different fusion channels



$X = \mathrm{I}$ or $X = \Psi$.

⟶ wave-function with non-abelian braiding properties

$$\Psi(z)_a = \prod_{i<j} z_{ij}^{1/4}\, F(z)_a = \prod_{i<j} z_{ij}^{1-g}\, F(z)_a$$

with $\quad F(z)_a \sim c_{a1} + c_{a2}\sqrt{z_{ij}} \quad$ for $\quad z_i \to z_j$.

- **non-polynomial eigenfunctions** of the CS Hamiltonian with non-abelian monodromy
- symmetric generalization of **hypergeometric functions** ~ **[Kaneko, 93; Forrester, 92]**
- can be represented as Coulomb integrals **[Dotsenko, Fateev, 84]**

Duality and the non-poynomial wave-functions

How to characterize an arbitrary excited **"partition"** λ (λ_i generically not integers) ?

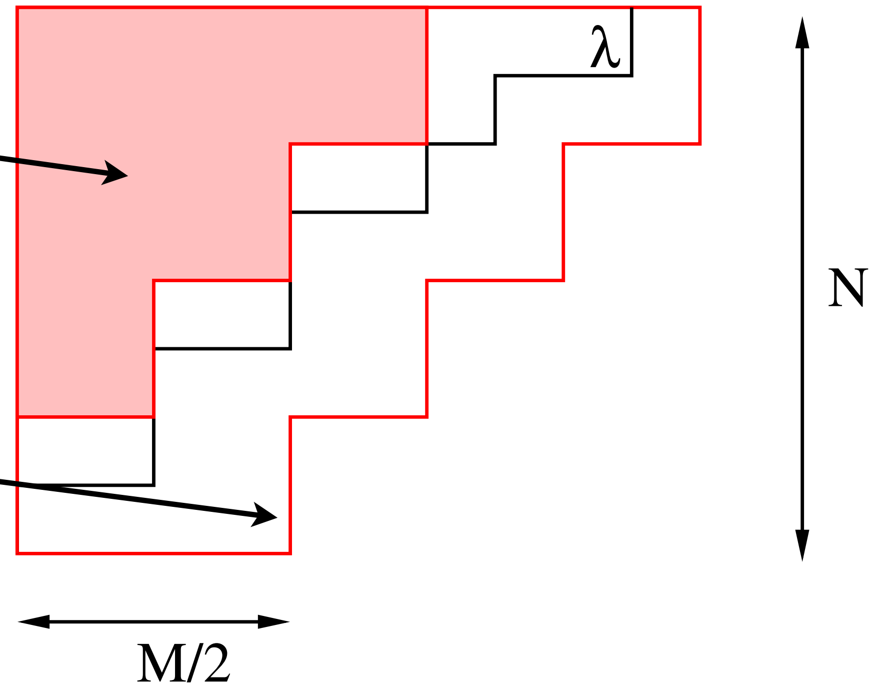
$$\mathcal{F}^{a,b}_{M,N}(w;z) = \sum_{\lambda} P^{\widetilde{\alpha},a}_{\lambda'}(w)\, P^{\alpha,b}_{\lambda}(z)$$

- **ground state** (M=0): $\langle \Phi_{12}(z_1)\cdots\Phi_{12}(z_N)\rangle_a \prod_{i<j} z_{ij}^{2h}$ ⟶ smallest "partition"

special conformal block with X=I: $z_1^{2h(N-2)} z_2^{2h(N-2)} z_3^{2h(N-4)} z_4^{2h(N-4)} \ldots z_{N-1}^0 z_N^0 + \ldots$

smallest "partition" $\lambda^0_{2i-1} = \lambda^0_{2i} \;=\; 2h(N-2i)$

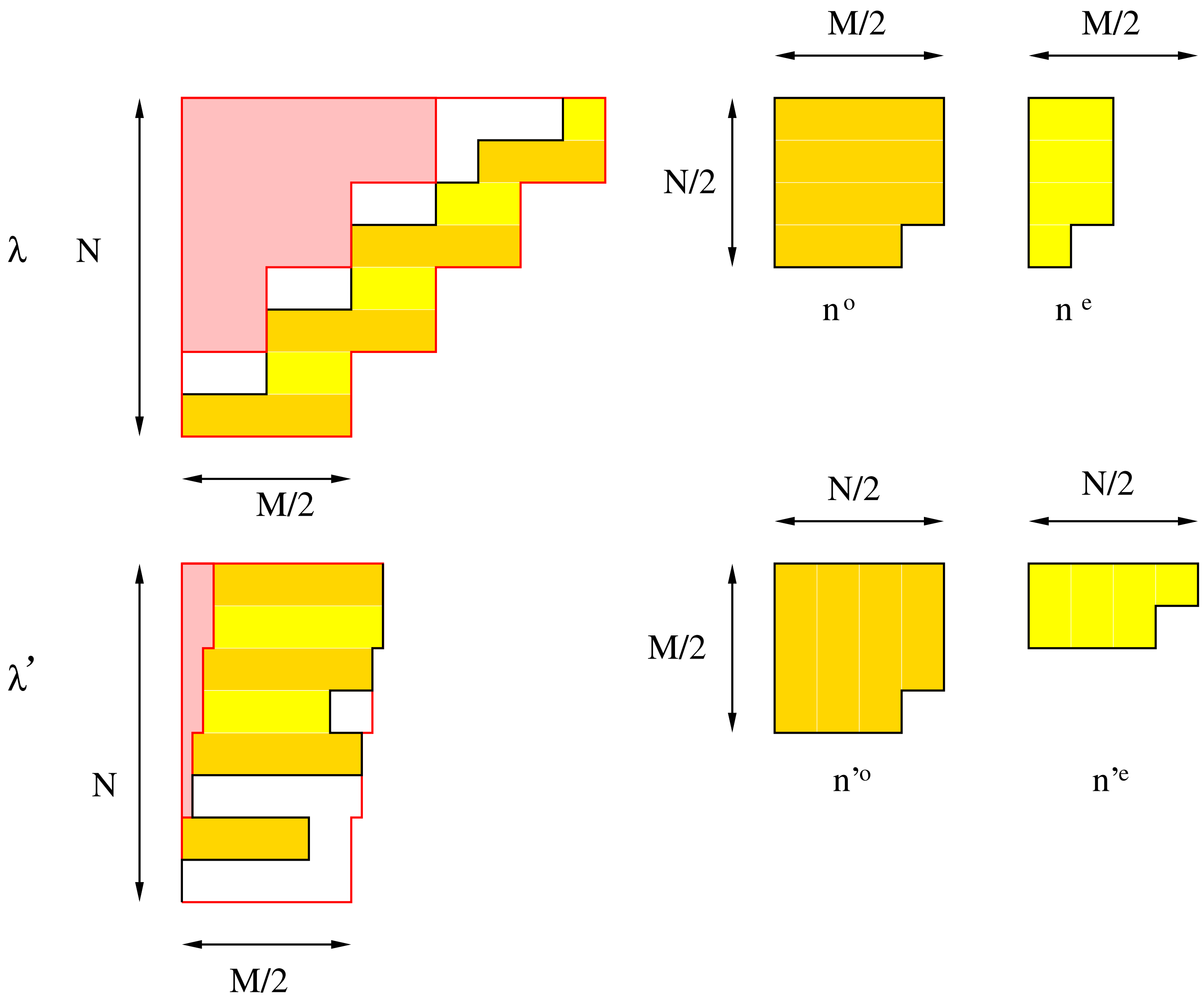
largest "partition" $\Lambda^0_i \;=\; \lambda^0_i + \dfrac{M}{2}$



- Ising: 2h=1 ⟶ λ is a partition satisfying the (2,2) admissibility condition $\lambda_i - \lambda_{i+2} \geq 2$

Duality and the non-poynomial wave-functions

An excited state λ is characterized by **two partitions** n_e and n_o (reminiscent of AGT conjecture):



for the partition n' **dual** to n:

$$b(n) \equiv 2\sum_i (i-1)n_i = \sum_i n'_j(n'_j - 1)$$

$$|n| \equiv \sum_i n_i = |n'|$$

$$\mathcal{E}^{\alpha}_{\lambda} = \sum_{i=1}^{N} \lambda_i[\lambda_i + (1-g)(N+1-2i)]$$

$$\mathcal{E}^{\tilde{\alpha}}_{\lambda'} = \sum_{j=1}^{M} \lambda'_j[\lambda'_j + (1-g^{-1})(M+1-2j)]$$

$$\mathcal{E}^{\alpha}_{\lambda} = [b(n'^o) + b(n'^e)] - g\,[b(n^o) + b(n^e)] + ((1-g)N - M + g)(|n^o| + |n^e|) + 2(g-1)|n^e| + \mathcal{E}^{\alpha}_{(M/2)^N}$$

$$\mathcal{E}^{\tilde{\alpha}}_{\lambda'} = [b(n^o) + b(n^e)] - \frac{1}{g}\,[b(n'^o) + b(n'^e)] + \frac{(2-g)M + 2g - 3}{g}(|n'^e| + |n'^o|) + \frac{2(g-1)}{g}|n'^o| + \mathcal{E}^{\tilde{\alpha}}_0$$

additive spectrum??

u(1) x Virasoro models

introduce a u(1) component $\longrightarrow$ electromagnetic current for FQHE $J(z) = i\partial\phi(z)$

Heisenberg algebra : $[a_n, a_m] = n\delta_{n+m,0}$

Virasoro algebra : $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$

Feigin-Fuchs representation : $L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} : b_{n-m}b_m : -\alpha_0(n+1)b_n$ $\quad 2\alpha_0 = \sqrt{\frac{2}{g}} - \sqrt{2g}$ $\quad c = 1 - 12\alpha_0^2$

Degenerate fields dressed by u(1) vertex operators :

$$V(z) \equiv \Phi_{12}(z)\, e^{i\sqrt{\frac{g}{2}}\phi(z)}\,, \qquad \widetilde{V}(w) \equiv \Phi_{21}(w)\, e^{i\frac{1}{\sqrt{2g}}\phi(w)}$$

Consider generic correlation functions :

$$f^+_\mu(z_1, z_2, \cdots, z_N) = \langle\mu|V(z_1)V(z_2)\cdots V(z_N)|P\rangle$$
$$f^-_\mu(z_1, z_2, \cdots, z_N) = \langle P|V(z_1)V(z_2)\cdots V(z_N)|\mu\rangle$$

$|\mu\rangle$: generic state (primary or descendant)

$|P\rangle$: primary state

Translate the CS action on states :

$$H^g_n f^\pm_\mu(z_1, z_2, \cdots, z_N) = \sum_\nu \left[I^\pm_{n+1}(g)\right]_{\mu,\nu} f^\pm_\nu(z_1, z_2, \cdots, z_N)$$

CS integrals of motions and the Hilbert space of u(1) x Virasoro

- **second-order** null vector condition: $\left(L_{-1}^2 - gL_{-2}\right) V = 0$

⟶ second order CS hamiltonian:

$$I_3^{(\pm)}(g) = 2(1-g)\sum_{m\geq 1} m a_{-m}a_m \pm \sqrt{2g}\sum_{m\neq 0} a_{-m}L_m \pm \sqrt{\frac{g}{2}}\left(\sum_{m,k\geq 1} a_{-m-k}a_m a_k + a_{-m}a_{-k}a_{m+k}\right)$$

- null vector condition at **level 3**: $\left(L_{-1} + 3\sqrt{g/2}a_{-1}\right)\left(L_{-1}^2 - gL_{-2}\right) V = 0$

⟶ third order CS hamiltonian:

$$I_n^{(\pm)}(g) \propto I_n^{(\mp)}(1/g)$$

$$\begin{aligned} I_4^{\pm}(g) = &-g\sum_{m>0} L_{-m}L_m \\ &-\frac{3}{2}g\sum_{m,p>0}\left(2L_{-p}a_{-m}a_{p+m} + 2a_{-m-p}a_m L_p + a_{-m}a_{-p}L_{p+m} + L_{-m-p}a_m a_p\right) \\ &\pm\frac{3}{2}\sqrt{2g}(g-1)\sum_{m>0} m(a_{-m}L_m + L_{-m}a_m) \pm 3\sqrt{2g}(g-1)\sum_{m,p>0} m(a_{-m}a_{-p}a_{m+p} + a_{-m-p}a_m a_p) \\ &-\frac{1}{2}gL_0^2 - 3gL_0\sum_{m>0} a_{-m}a_m + \sum_{m\geq 1}\left[\frac{1}{2}(9g-5-5g^2)m^2 - \frac{1}{2}(g-1)^2\right] a_{-m}a_m \\ &-\frac{g}{8}\sum_{\substack{m_1+m_2+m_3+m_4=0 \\ m_i\neq 0}} :a_{m_1}a_{m_2}a_{m_3}a_{m_4}: \end{aligned}$$

[Alba, Fateev, Litvinov, Tarnopolsky,10]
for Liouville $g \to -g$

Jack polynomials and the Hilbert space of u(1) x Virasoro

- rotate the boson basis: $c_m = \frac{1}{\sqrt{2}}(a_m + b_m)\ , \qquad \widetilde{c}_m = \frac{1}{\sqrt{2}}(a_m - b_m)$ **[AFLT, 10; Belavin and Belavin, 11]**

- introduce the one-component bosonised CS Hamiltonians:

$$\mathcal{I}_3^{\pm}(c;g) = (1-g)\sum_{m>0} m c_{-m}c_m \pm \sqrt{g}\sum_{m,k>0}(c_{-m-k}c_m c_k + c_{-m}c_{-k}c_{m+k})$$

[Jevicki, 91]
[Awata, Matsuo, Odake, Shiraishi, 95]

$$\begin{aligned}\mathcal{I}_4^{\pm}(c;g) &= \pm\left(\frac{3g}{2} - g^2 - 1\right)\sum_{m>0} m^2 c_{-m}c_m - \frac{g}{4}\sum_{\substack{m_1+m_2+m_3+m_4=0\\ m_i\neq 0}} :c_{m_1}c_{m_2}c_{m_3}c_{m_4}: \pm\\ &\pm\ 3\sqrt{g}(g-1)\sum_{m,l>0} m(c_{-m-l}c_m c_l + c_{-m}c_{-l}c_{m+l})\ .\end{aligned}$$

- classical limit $g \to 0$, $\quad v = \sqrt{g}\partial\phi$ and $\mathcal{I}_n \to g\mathcal{I}_n$ $\longrightarrow$ Benjamin-Ono hierarchy:

$$\begin{aligned}\mathcal{I}_2 &= \int dx\ \frac{1}{2}v^2\ ,\\ \mathcal{I}_3 &= \int dx\left(\frac{1}{3}v^3 + \frac{1}{2}vH(v_x)\right)\ ,\\ \mathcal{I}_4 &= \int dx\left(\frac{1}{4}v^4 + \frac{1}{4}v_x^2 + \frac{3}{4}v^2H(v_x)\right)\end{aligned}$$

$H(f)$ is the Hilbert transform of the function f.

$$v_x = \partial_x v$$

CS ⟷ BO: **[Abanov, Wiegmann, 05]**

Jack polynomials and the Hilbert space of u(1) x Virasoro

At g=1 the CS Hamiltonian is a sum of two copies of one-component CS models (up to zero modes): **[Belavin and Belavin, 11]**

$$I_3^+(1) \;=\; \mathcal{I}_3(c) + \mathcal{I}_3(\widetilde{c}) + \sqrt{2}b_0(L_0(c) - L_0(\widetilde{c}) - a_0 b_0)$$

$$I_4^+(1) = \mathcal{I}_4(c) + \mathcal{I}_4(\widetilde{c}) - b_0 D(c, \widetilde{c}, b_0)$$

The states can be constructed with the help of Schur polynomials:

$$|n^o, n'^e; q\rangle = S_{n^o}(c)S_{n^e}(\widetilde{c})|q\rangle + S_{n^o}(\widetilde{c})S_{n^e}(c)|-q\rangle$$

$$c_{-n} \sim p_n = \sum_i x_i^n$$

$$b_0|q\rangle = q|q\rangle$$

At arbitrary g there is an interacting term with triangular structure:

$$I_3^+(g) \;=\; \mathcal{I}_3^+(c;g) + \mathcal{I}_3^+(\widetilde{c};g) + (\sqrt{2g}b_0 + g - 1)(L_0(c) - L_0(\widetilde{c})) + 2(1-g)\sum_{m>0} m c_{-m}\widetilde{c}_m$$

also

[Maulik, Okounkov, unpublished]

[Shou, Wu, Yu, 11]

⟶ **additive spectrum!** : $E^{\pm}_{3;\,n^o,n^e}(g) = e^{\pm}_{3,n^o}(g) + e^{\pm}_{3,n^e}(g) \pm (g-1)(|n^o| - |n^e|)$

triangular structure: $|n^o, n^e; q\rangle = J^{1/g}_{n^o}(c)\ J^{1/g}_{n^e}(\widetilde{c})\ |q\rangle + \dots$

- **example:** states at level 2 in the identity module (3 states out of 5 by symmetry reasons)

$$|n^o, n^e; q\rangle = |n^e, n^o; -q\rangle \qquad \to \qquad |n^o, n^e; 0\rangle = |n^e, n^o; 0\rangle$$

$$\begin{aligned}
|[2],[\emptyset] > &= \left[(2-3g)a_{-2} + \left(\frac{3}{2} - \frac{1}{g}\right)\sqrt{2g}a^2_{-1} + \sqrt{2g}L_{-2}\right]|N > \\
|[1,1],[\emptyset] > &= \left[(3-2g)a_{-2} + \left(\frac{3}{2} - g\right)\sqrt{2g}a^2_{-1} + \sqrt{2g}L_{-2}\right]|N > \\
|[1],[1] > &= \left[(1-g)a_{-2} - \frac{1}{2}\sqrt{2g}a^2_{-1} + \sqrt{2g}L_{-2}\right]|N >
\end{aligned}$$

WA_{k-1} algebras

The same construction extends to WA_{k-1} algebras

k-1 bosons x u(1) component: ⟶ k- component CS Hamiltonian

$$I_3^{(\pm)}(g) = 2(1-g)\sum_{m\geq 1} m a_{-m}a_m \pm \sqrt{2g}\sum_{m\geq 1}(a_{-m}L_m + L_{-m}a_m) \pm \sqrt{\frac{g}{2}}\left(\sum_{m,k\geq 1} a_{-m-k}a_m a_k + a_{-m}a_{-k}a_{m+k}\right) \pm \widetilde{W}_0$$

c^j_m: k copies of mutually commuting bosons

$$I_3^{+}(g) = \sum_{j=1}^{k}\mathcal{I}^{\pm}(c^j;g) + 2(1-g)\sum_{j<l}\sum_{m\geq 1} m : c^j_{-m}c^l_m : + (1-g)\sum_j d_j L_0(c^j) + \text{zero modes}$$

⟶ additive spectrum depending on **k partitions** (~ AGT conjecture for U(k) theories)

[see also Fateev, Litvinov, this morning

arXiv1109.4042]

Conclusions

- We have learned how to characterize the states of the Vir x H CFT, or WA_{k-1} x H, in terms of CS integrals of motion
- AFLT: this basis gives an efficient way to compute matrix elements of the fields (representation of the conformal blocks)
- Similar structure in the FQHE (different physics)

open questions

- Theory of non-polynomial CS eigenfunctions?
- How to systematically generate the integrals of motion (transfer matrix?) in CFT? **[Maulik, Okounkov, unpublished]** in 4d gauge theory context
- Relation with the integrable structure uncovered by **[Bazhanov, Lukyanov, Zamolodchikov, 94-98]** (no Heisenberg factor)?

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