A random growth model in the perfect matching problem

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(arXiv:1102.1558)

Random Processes, Conformal Field Theory & Integrable Systems
19–23 September 2011, Laboratoire J.-V. Poncelet

Minimum-weight perfect matching on a line

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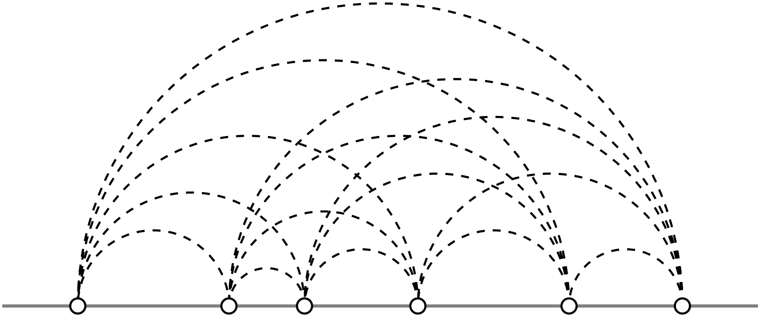

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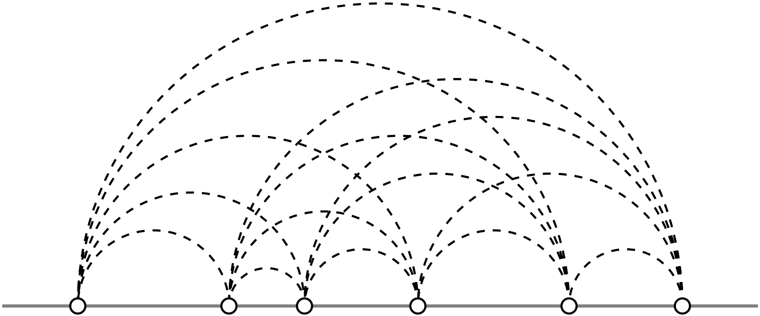

Arc weights ≈ distances:

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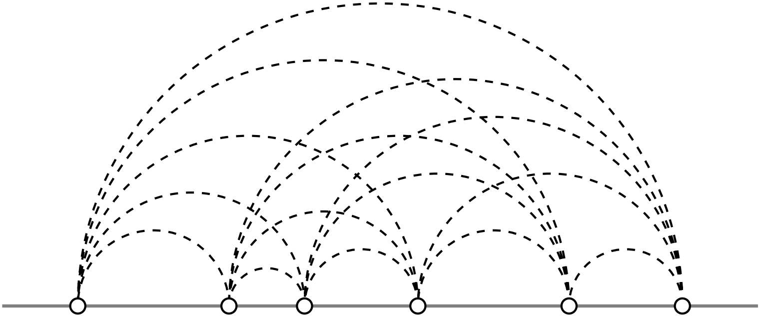

Arc weights ≈ distances:

- positive (*not essential*)
- increasing with $|x_i - x_j|$
- homogeneous (w_{x_i,x_j} *depends on* $x_i - x_j$)
- triangle inequality (w_{x_i,x_j} *concave in* $x_i - x_j$)

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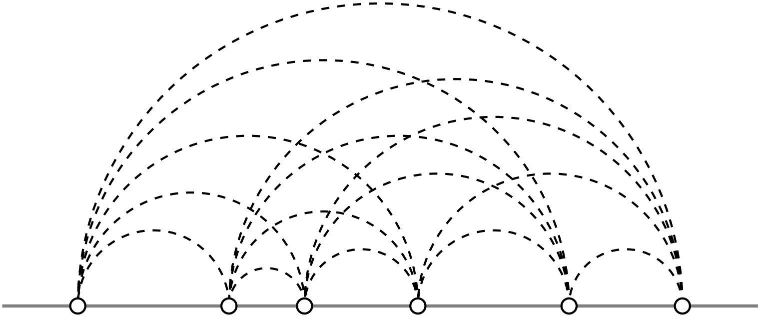

Arc weights ≈ distances:

$$w_{x_i,x_j} = |x_i - x_j|^{\alpha},\ 0 < \alpha < 1, \quad \text{or} \quad w_{x_i,x_j} = \log|x_i - x_j|$$

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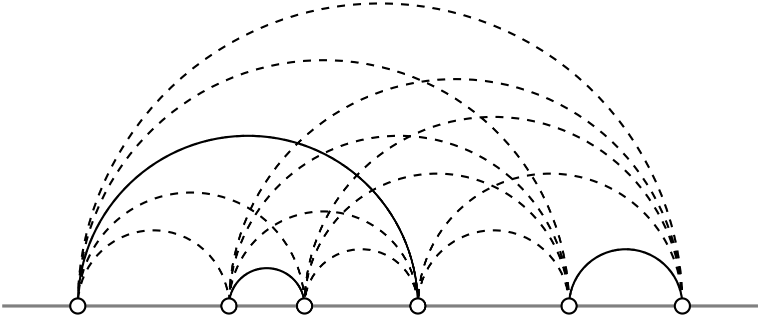

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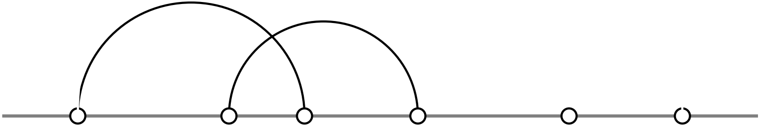

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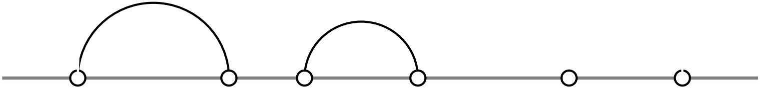

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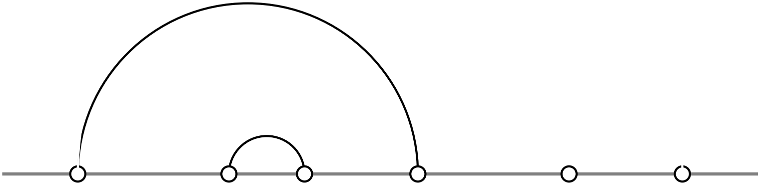

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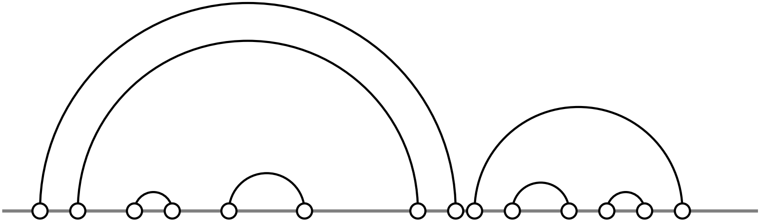

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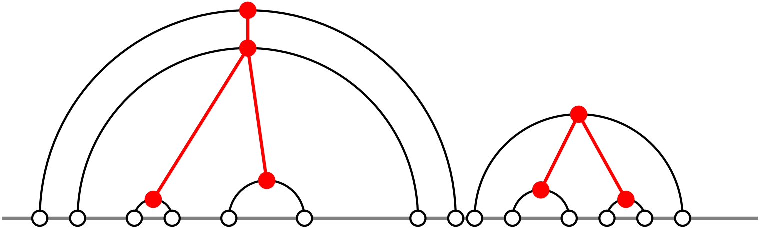

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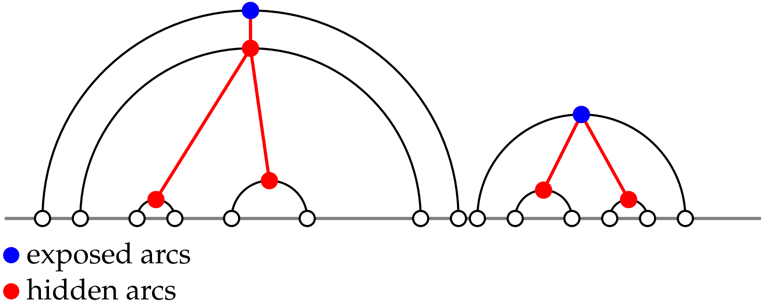

Observation II: persistence of hidden arcs

— Add a few pairs of vertices outside an already existing configuration; how will the optimal matching change?

— All hidden arcs are preserved and remain hidden.

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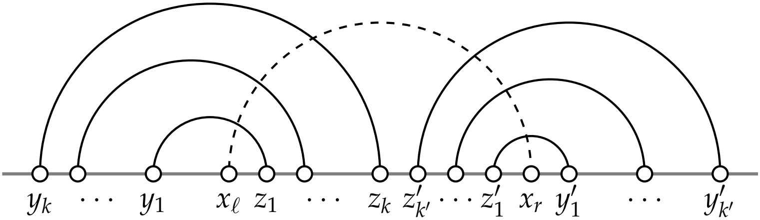

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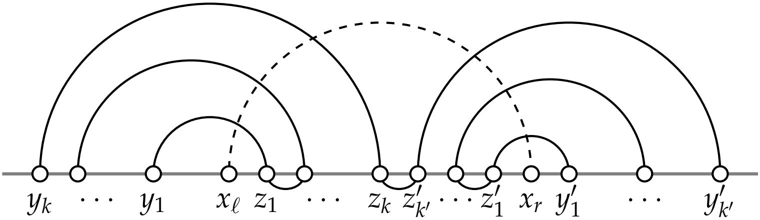

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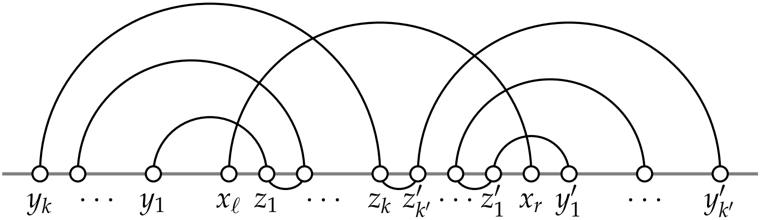

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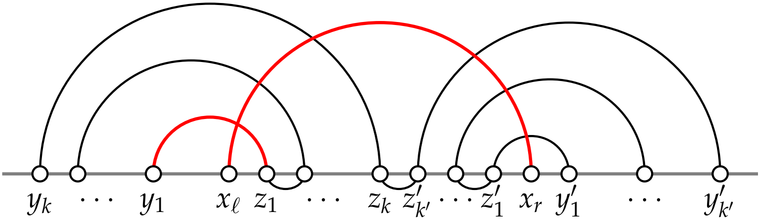

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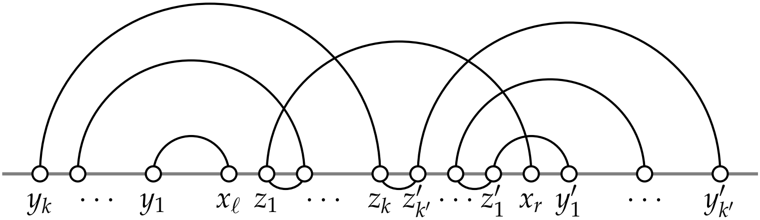

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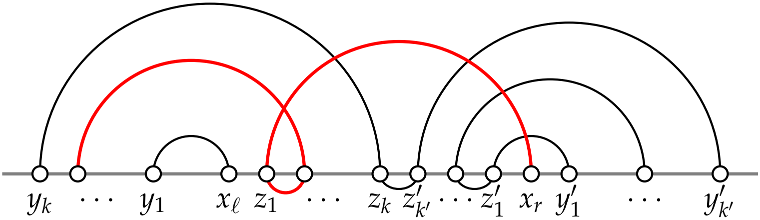

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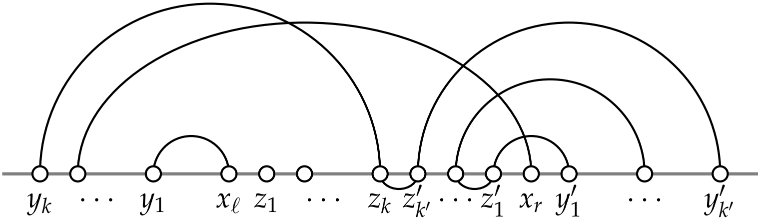

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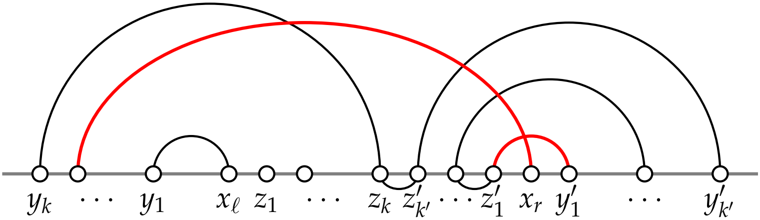

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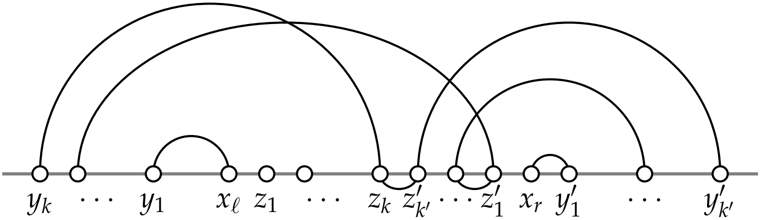

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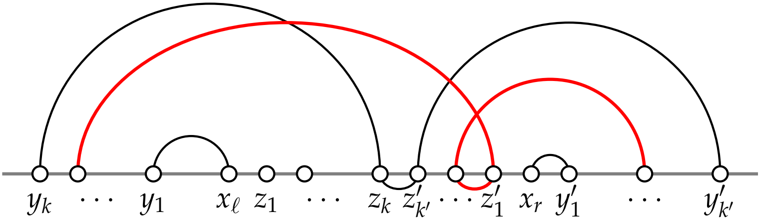

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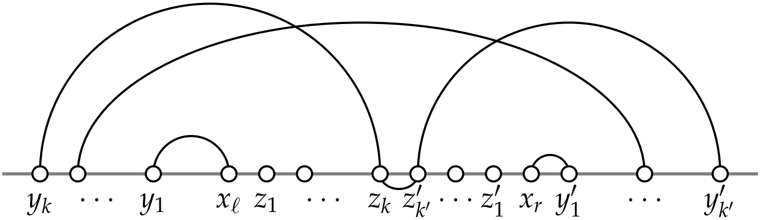

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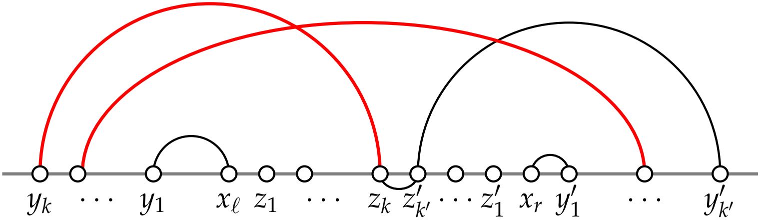

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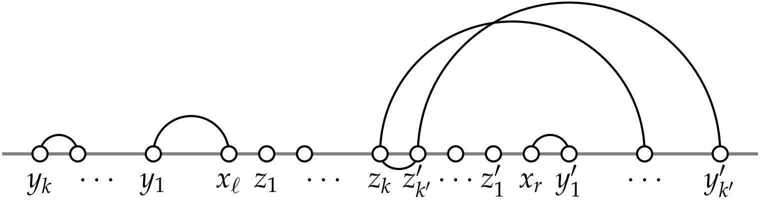

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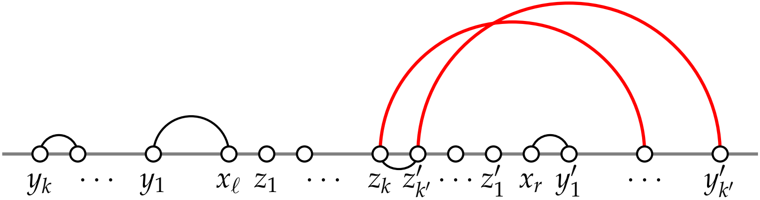

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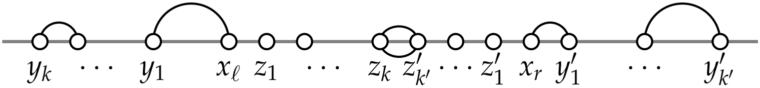

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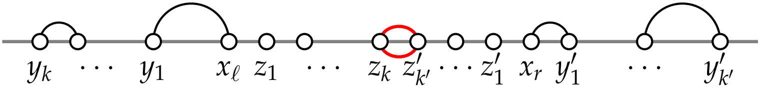

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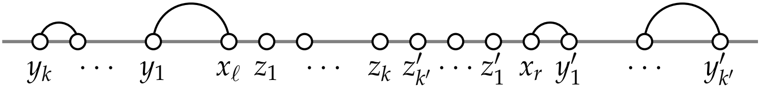
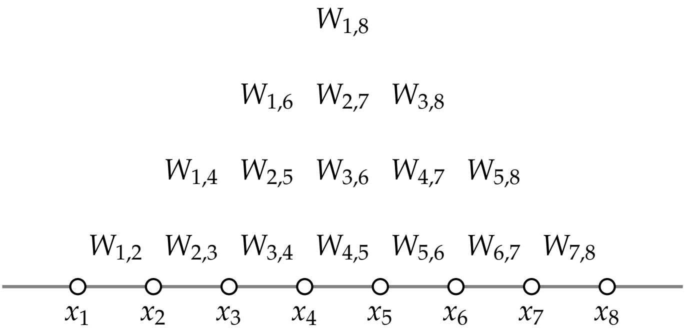


Observation III: recursion for optimal matching



Goal: Find a simple recursion relation for $W_{i,j}$'s

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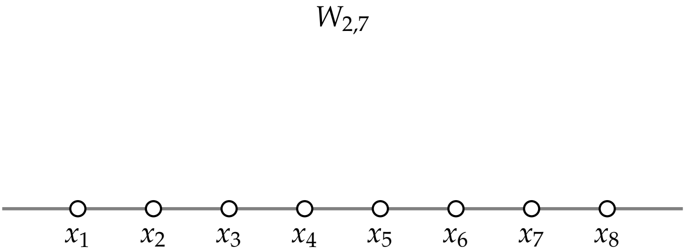

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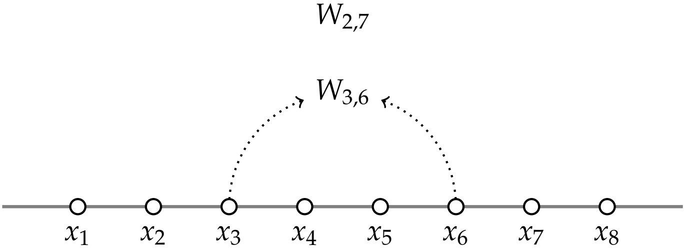

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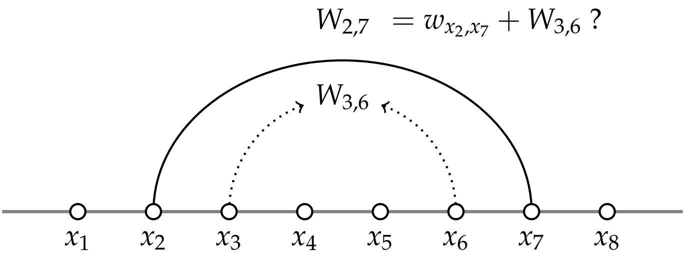

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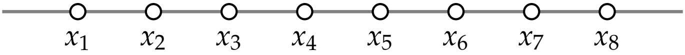
$W_{2,7}$



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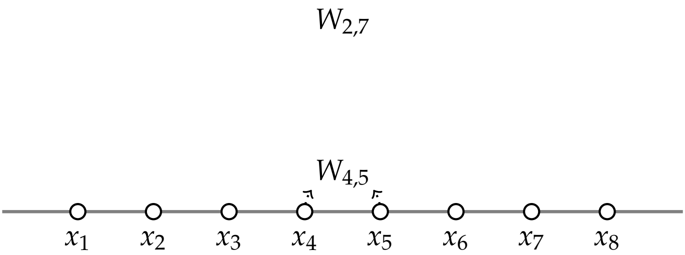

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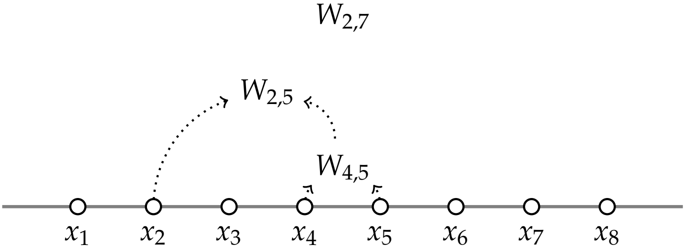

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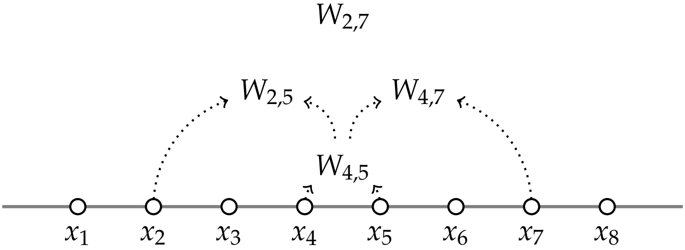

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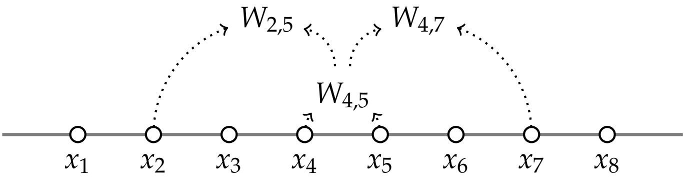
$$W_{2,7} \;= W_{2,5} + W_{4,7} - W_{4,5}\ ?$$



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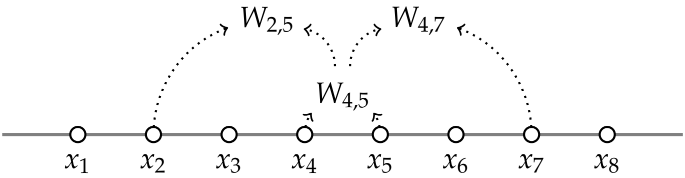

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$$W_{i,j} = \min\left[w_{x_i,x_j} + W_{i+1,j-1},\ W_{i,j-2} + W_{i+2,j} - W_{i+2,j-2}\right]$$

Remark II: hyperbolicity revealed

$$W_{i,j} = \min\left[w_{x_i,x_j} + W_{i+1,j-1},\ W_{i,j-2} + W_{i+2,j} - W_{i+2,j-2}\right]$$

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$$z = \frac{i+j}{2} \cdot c\tau, \quad t = \frac{j-i}{2} \cdot \tau$$

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$$\begin{aligned} 0 = \min\Big[& w(z-ct, z+ct) + W(z, t-\tau) - W(z,t), \\ & W(z-c\tau, t-\tau) + W(z+c\tau, t-\tau) - 2W(z, t-\tau) \\ & \qquad - W(z,t) - W(z, t-2\tau) + 2W(z, t-\tau)\Big] \end{aligned}$$

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$$0 = \min\Big[\omega(z-ct, z+ct) - \partial_t W(z,t),$$
$$c^2 \partial_z^2 W(z,t) - \partial_t^2 W(z,t)\Big]$$

$$\omega = \lim_{\tau \to 0} \frac{1}{\tau} w$$

The end