Quantum Painleve-Calogero Correspondence

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(Based on joint work with A. Zotov)

Painleve equations

$$\mathrm{P_I}: \quad \partial_T^2 q = 6q^2 + T$$

$$\mathrm{P_{II}}: \quad \partial_T^2 q = 2q^3 + Tq + \alpha$$

$$\mathrm{P_{III}}: \quad \partial_T^2 q = \frac{(\partial_T q)^2}{q} - \frac{\partial_T q}{T} + \frac{\alpha q^2 + \beta}{T} + \gamma q^3 + \frac{\delta}{q}$$

$$\mathrm{P_{IV}}: \quad \partial_T^2 q = \frac{(\partial_T q)^2}{2q} + \frac{3}{2} q^3 + 4Tq^2 + 2(T^2 - \alpha) q + \frac{\beta}{q}$$

$$\mathrm{P_V}: \quad \partial_T^2 q = \left(\frac{1}{2q} + \frac{1}{q-1}\right)(\partial_T q)^2 - \frac{\partial_T q}{T}$$

$$+ \frac{q(q-1)^2}{T^2}\left(\alpha + \frac{\beta}{q^2} + \frac{\gamma T}{(q-1)^2} + \frac{\delta T^2(q+1)}{(q-1)^3}\right)$$

$$\mathrm{P_{VI}}: \quad \partial_T^2 q = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-T}\right)(\partial_T q)^2$$

$$- \left(\frac{1}{T} + \frac{1}{T-1} + \frac{1}{q-T}\right)\partial_T q$$

$$+ \frac{y(y-1)(y-T)}{T^2(T-1)^2}\left(\alpha + \frac{\beta T}{y^2} + \frac{\gamma(T-1)}{(y-1)^2} + \frac{\delta T(T-1)}{(y-T)^2}\right)$$

Compatibility of linear problems

$$\begin{cases} \partial_X \vec{\Psi} = \mathcal{L}(X,T)\vec{\Psi} \\ \\ \partial_T \vec{\Psi} = \mathcal{M}(X,T)\vec{\Psi} \end{cases} \qquad \vec{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Zero curvature condition

$$\partial_X \mathcal{M} - \partial_T \mathcal{L} + [\mathcal{M},\, \mathcal{L}] = 0$$

is equivalent to Painleve equation

Example (Painleve I)

$$\mathcal{L}(X,T) = \begin{pmatrix} \partial_T q & X - q \\ X^2 + Xq + q^2 + \frac{1}{2}T & -\partial_T q \end{pmatrix}$$

$$\mathcal{M}(X,T) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2}X + q & 0 \end{pmatrix}$$

Hamiltonian structure

There is a time-dependent Hamiltonian H(p, q; T) such that Painleve equation is equivalent to

$$\partial_T q = \frac{\partial \mathcal{H}}{\partial p}, \qquad \partial_T p = -\frac{\partial \mathcal{H}}{\partial q}$$

H(p, q; T) is a polynomial in p,q (K.Okamoto)

Painleve-Calogero correspondence

(P.Painleve, 1906; Yu. Manin, 1996;
A.Levin and M.Olshanetsky, 1997;
K.Takasaki, 2001)

There exists a change of variables

$$(q,T) \longrightarrow (u,t)$$

such that the Painleve equation acquires the form of the Newton equation

$$\ddot{u} = -\partial_u V(u,t)$$

with Hamiltonian

$$H(p,q;t) = \frac{p^2}{2} + V(u,t)$$

	$q(u,t)$	$T(t)$
$\mathrm{P_I}$	u	t
$\mathrm{P_{II}}$	u	t
$\mathrm{P_{IV}}$	u^2	t
$\mathrm{P_{III}}$	e^{2u}	e^t
$\mathrm{P_V}$	$\coth^2 u$	e^{2t}
$\mathrm{P_{VI}}$	$\frac{\wp(u)-\wp(\omega_1)}{\wp(\omega_2)-\wp(\omega_1)}$	$\frac{\wp(\omega_3)-\wp(\omega_1)}{\wp(\omega_2)-\wp(\omega_1)}$

$$\mathrm{P_I}: \quad V(x,t) = -\frac{x^3}{2} - \frac{tx}{4}$$

$$\mathrm{P_{II}}: \quad V(x,t) = -\frac{1}{2}\left(x^2 + \frac{t}{2}\right)^2 + \alpha x$$

$$\mathrm{P_{IV}}: \quad V(x,t) = -\frac{x^6}{8} - \frac{tx^4}{2} - \frac{1}{2}\left(t^2 - \alpha\right)x^2 + \frac{\beta}{4x^2}$$

$$\mathrm{P_{III}}: \quad V(x,t) = -\alpha e^{2x+t} - \beta e^{-2x+t} - \gamma e^{4x+2t} - \delta e^{-4x+2t}$$

$$\mathrm{P_V}: \quad V(x,t) = -\frac{\alpha}{\sinh^2 x} - \frac{\beta}{\cosh^2 x}$$

$$+ \frac{\gamma e^{2t}}{2}\cosh(2x) + \frac{\delta e^{4t}}{8}\cosh(4x)$$

$$\mathrm{P_{VI}}: \quad V(x,t) = -\sum_{k=0}^{3} \nu_k \wp(x+\omega_k)$$

$$\wp(x) = \wp(x|\,1,\tau), \qquad \tau = 2\pi i t$$

$$\omega_0 = 0,\ \omega_1 = \frac{1}{2},\ \omega_2 = \frac{1}{2}(1+\tau),\ \omega_3 = \frac{1}{2}\tau$$

$$\nu_0 = \alpha,\ \nu_1 = -\beta,\ \nu_2 = \gamma,\ \nu_3 = -\delta + \frac{1}{2}$$

Resembles a particular case of elliptic Calogero or Inozemtzev model

Quantum Painleve-Calogero correspondence

emerges when we apply this change of variables to the auxiliary linear problems and supplement it by an appropriate change of the spectral parameter

$$X \longrightarrow x\,,\; X = X(x,t)$$

$$\begin{cases} \partial_X \vec{\Psi} = \mathcal{L}(X,T)\vec{\Psi} \\ \partial_T \vec{\Psi} = \mathcal{M}(X,T)\vec{\Psi} \end{cases} \longrightarrow \begin{cases} \partial_x \Psi = L(x,t)\Psi \\ \partial_t \Psi = M(x,t)\Psi \end{cases}$$

$$\vec{\Psi} \longrightarrow \Psi = \Omega\vec{\Psi} \longrightarrow \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Example (Painleve I)

$$L(x,t) = \begin{pmatrix} \dot{u} & x - u \\ x^2 + xu + u^2 + \frac{1}{2}t & -\dot{u} \end{pmatrix}$$

$$M(x,t) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2}x + u & 0 \end{pmatrix}$$

	$q(u,t)$	$T(t)$	$X(x,t)$
P_{I}	u	t	x
P_{II}	u	t	x
P_{IV}	u^2	t	x^2
P_{III}	e^{2u}	e^t	e^{2x}
P_{V}	$\coth^2 u$	e^{2t}	$\cosh^2 x$
P_{VI}	$\frac{\wp(u)-\wp(\omega_1)}{\wp(\omega_2)-\wp(\omega_1)}$	$\frac{\wp(\omega_3)-\wp(\omega_1)}{\wp(\omega_2)-\wp(\omega_1)}$	$\frac{\wp(x)-\wp(\omega_1)}{\wp(\omega_2)-\wp(\omega_1)}$

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\begin{cases} \partial_x \psi_1 = a\psi_1 + b\psi_2 \\ \partial_x \psi_2 = c\psi_1 + d\psi_2 \end{cases} \qquad \begin{cases} \partial_t \psi_1 = A\psi_1 + B\psi_2 \\ \partial_t \psi_2 = C\psi_1 + D\psi_2 \end{cases}$$

$$\psi_2 = \frac{\partial_x \psi_1 - a\psi_1}{b} = \frac{\partial_t \psi_1 - A\psi_1}{B}$$

$$\psi_1 \equiv \Psi$$

Painleve equation is represented as compatibility condition for two scalar linear problems of the form

$$\begin{cases} \left(\frac{1}{2}\partial_x^2 - \frac{b_x}{2b}\partial_x + W(x,t)\right)\Psi = 0 \\ \partial_t \Psi = \left(\frac{1}{2}\partial_x^2 + V^{(q)}(x,t)\right)\Psi \end{cases}$$

$$W(x,t) = V^{(q)}(x,t) - H(\dot{u},u,t) - \frac{2b_t - b_{xx}}{4b}$$

$$H(\dot{u},u,t) = \frac{\dot{u}^2}{2} + V^{(cl)}(u,t)$$

$$b(x,t) = L_{12}(x,t) \qquad\qquad b(u,t) = 0$$

(A similar observation was made by B.Suleimanov, 1994)

	$q(u,t)$	$T(t)$	$X(x,t)$	$b(x,t) = L_{12}(x,t)$
P_{I}	u	t	x	$x-u$
P_{II}	u	t	x	$x-u$
P_{IV}	u^2	t	x^2	x^2-u^2
P_{III}	e^{2u}	e^t	e^{2x}	$2e^{t/2}\sinh(x-u)$
P_{V}	$\coth^2 u$	e^{2t}	$\cosh^2 x$	$2e^t\sinh(x-u)\sinh(x+u)$
P_{VI}	$\frac{\wp(u)-\wp(\omega_1)}{\wp(\omega_2)-\wp(\omega_1)}$	$\frac{\wp(\omega_3)-\wp(\omega_1)}{\wp(\omega_2)-\wp(\omega_1)}$	$\frac{\wp(x)-\wp(\omega_1)}{\wp(\omega_2)-\wp(\omega_1)}$	$\vartheta_1(x-u)\vartheta_1(x+u)h(u,t)$

For P_{I}, P_{II}, P_{III} $V^{(q)}(x,t) = V^{(cl)}(x,t)$!

For P_{IV}, P_{V}, P_{VI} $V^{(q)}(x,t)$ differs from $V^{(cl)}(x,t)$ by a shift of parameters

$$(\tilde{\alpha}, \tilde{\beta}) = (\alpha, \beta + \tfrac{1}{2}) \quad \text{for } P_{\mathrm{IV}},$$

$$(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = (\alpha - \tfrac{1}{8}, \beta + \tfrac{1}{8}, \gamma, \delta) \quad \text{for } P_{\mathrm{V}}$$

$$(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = (\alpha - \tfrac{1}{8}, \beta + \tfrac{1}{8}, \gamma - \tfrac{1}{8}, \delta + \tfrac{1}{8}) \quad \text{for } P_{\mathrm{VI}}$$

("quantum corrections")

The non-stationary Schrodinger (or Fokker-Planck) equation

$$\partial_t \Psi = \left(\frac{1}{2}\partial_x^2 + V^{(q)}(x,t)\right)\Psi$$

is one of the equations of the Lax pair for Painleve, the one defining the time evolution, and, simultaneously, **quantization** of the Painleve equation!

Having this equation, one does not need the second element of the Lax pair because it already reproduces Painleve by classical limit.

For Painleve VI with special choice of parameters we obtain the non-stationary Lame equation

$$\partial_t \Psi = \left(\frac{1}{2}\partial_x^2 + \nu \wp(2x|1, 2\pi i t)\right)\Psi$$

which appears in conformal field theory and theory of XYZ spin chain.

Quantum Painleve-Calogero correspondence: conclusion

$$\partial_t \Psi = \left(\frac{1}{2}\partial_x^2 + V(x,t)\right)\Psi$$

Painleve side

Auxiliary linear problem for Painleve

Calogero side

Quantum Calogero-like model in a non-stationary state

Quantization is equivalent to linearization (passing to auxiliary linear problem)