The Brauer-Siegel and Tsfasman-Vlăduţ Theorems for Almost Normal Extensions of Number Fields

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Abstract

The classical Brauer-Siegel theorem states that if k runs through the sequence of normal extensions of $\mathbb{Q}$ such that $n_k/\log|D_k| \to 0$, then $\log h_k R_k/\log\sqrt{|D_k|} \to 1$. First, in this paper we obtain the generalization of the Brauer-Siegel and Tsfasman-Vlăduţ theorems to the case of almost normal number fields. Second, using the approach of Hajir and Maire, we construct several new examples concerning the Brauer-Siegel ratio in asymptotically good towers of number fields. These examples give smaller values of the Brauer-Siegel ratio than those given by Tsfasman and Vlăduţ.

1. Introduction

Let K be an algebraic number field of degree $n_K = [K:\mathbb{Q}]$ and discriminant D_K. We define the genus of K as $g_K = \log\sqrt{D_K}$. By h_K we denote the class-number of K, R_K denotes its regulator. We call a sequence $\{K_i\}$ of number fields a family if K_i is non-isomorphic to K_j for $i \neq j$. A family is called a tower if also $K_i \subset K_{i+1}$ for any i. For a family of number fields we consider the limit

$$\mathrm{BS}(\mathcal{K}) := \lim_{i\to\infty} \frac{\log h_{K_i} R_{K_i}}{g_{K_i}}.$$

The classical Brauer-Siegel theorem, proved by Brauer (see [1]), states that for a family $\mathcal{K} = \{K_i\}$ we have $\mathrm{BS}(\mathcal{K}) = 1$ if the family satisfies two conditions:

(i) $\lim\limits_{i\to\infty} \frac{n_{K_i}}{g_{K_i}} = 0$;

(ii) either the generalized Riemann hypothesis (GRH) holds, or all the fields K_i are normal over $\mathbb{Q}$.

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We call a number field almost normal if there exists a finite tower of number fields $\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_m = K$ such that all the extensions K_i/K_{i-1} are normal. Weakening the condition (ii), we prove the following generalization of the classical Brauer-Siegel theorem to the case of almost normal number fields:

Theorem 1. *Let $\mathcal{K} = \{K_i\}$ be a family of almost normal number fields for which $n_{K_i}/g_{K_i} \to 0$ as $i \to \infty$. Then we have* $\mathrm{BS}(\mathcal{K}) = 1$.

It was shown by M. A. Tsfasman and S. G. Vlăduţ that, taking in account non-archimedian places, one may generalize the Brauer-Siegel theorem to the case of extensions where the condition (i) does not hold.

For a prime power q we set

$$N_q(K_i) := |\{v \in P(K_i) : \mathrm{Norm}(v) = q\}|,$$

where $P(K_i)$ is the set of non-archimedian places of K_i. We also put $N_{\mathbb{R}}(K_i) = r_1(K_i)$ and $N_{\mathbb{C}}(K_i) = r_2(K_i)$, where r_1 and r_2 stand for the number of real and (pairs of) complex embeddings.

We consider the set $A = \{\mathbb{R}, \mathbb{C}; 2, 3, 4, 5, 7, 8, 9, \ldots\}$ of all prime powers plus two auxiliary symbols $\mathbb{R}$ and $\mathbb{C}$ as the set of indices. A family $\mathcal{K} = \{K_i\}$ is called asymptotically exact if and only if for any $\alpha \in A$ the following limit exists:

$$\phi_\alpha = \phi_\alpha(\mathcal{K}) := \lim_{i \to \infty} \frac{N_\alpha(K_i)}{g_{K_i}}.$$

We call an asymptotically exact family $\mathcal{K}$ asymptotically good (respectively, bad) if there exists $\alpha \in A$ with $\phi_\alpha > 0$ (respectively, $\phi_\alpha = 0$ for any $\alpha \in A$). The condition on a family to be asymptotically bad is, in the number field case, obviously equivalent to the condition (i) in the classical Brauer-Siegel theorem. For an asymptotically good tower of number fields the following generalization of the Brauer-Siegel theorem was proved in [11]:

Theorem 2 (Tsfasman-Vlăduţ Theorem, see [11], Theorem 7.3). *Assume that for an asymptotically good tower $\mathcal{K}$ any of the following conditions is satisfied:*

- *GRH holds*

- *All the fields K_i are almost normal over $\mathbb{Q}$.*

Then the limit $\mathrm{BS}(\mathcal{K}) = \lim\limits_{i \to \infty} \frac{\log h_{K_i} R_{K_i}}{g_{K_i}}$ *exists and we have:*

$$\mathrm{BS}(\mathcal{K}) = 1 + \sum_q \phi_q \log \frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi, \qquad (1)$$

the sum beeing taken over all prime powers q.

For an asymptotically bad tower of number fields we have $\phi_{\mathbb{R}} = 0$ and $\phi_{\mathbb{C}} = 0$ as well as $\phi_q = 0$ for all prime powers q, so the right hand side of the formula (1) equals to one. We also notice that the condition on a family to be asymptotically bad is equivalent to $\lim\limits_{i \to \infty} \frac{n_{K_i}}{g_{K_i}} = 0$. So, combining our theorem 1 with the theorem 2 we get the following corollary:

Corollary 3. *For any tower $\mathcal{K} = \{K_i\}$, $K_1 \subset K_2 \subset \ldots$ of almost normal number fields the limit $\mathrm{BS}(\mathcal{K})$ exists and we have:*

$$\mathrm{BS}(\mathcal{K}) = \lim_{i \to \infty} \frac{\log(h_i R_i)}{g_i} = 1 + \sum_q \phi_q \log \frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi,$$

the sum beeing taken over all prime powers q.

In [11] bounds on the ratio $\mathrm{BS}(\mathcal{K})$ were given, together with examples showing that the value of $\mathrm{BS}(\mathcal{K})$ may be different from 1. We corrected some of these erroneous bounds and managed to precise a few of the estimates in the examples. Also, using the infinite tamely ramified towers, found by Hajir and Maire (see [3]), we get (under GRH) new examples, both totaly complex and totally real, with the values of $\mathrm{BS}(\mathcal{K})$ smaller than those of totally real and totally complex examples of [11]. The result is as follows:

Theorem 4.   1. *Let $k = \mathbb{Q}(\xi)$, where ξ is a root of $f(x) = x^6 + x^4 - 4x^3 - 7x^2 - x + 1$, $K = k(\sqrt{\xi^5 - 467\xi^4 + 994\xi^3 - 3360\xi^2 - 2314\xi + 961})$. Then K is totally complex and has an infinite tamely ramified 2-tower $\mathcal{K}$, for which, under GRH, we have:*

$$\mathrm{BS}_{\mathrm{lower}} \leq \mathrm{BS}(\mathcal{K}) \leq \mathrm{BS}_{\mathrm{upper}},$$

where $\mathrm{BS}_{\mathrm{lower}} \approx 0.56498\ldots$, $\mathrm{BS}_{\mathrm{upper}} \approx 0.59748\ldots$.

2. *Let $k = \mathbb{Q}(\xi)$, where ξ is a root of $f(x) = x^6 - x^5 - 10x^4 + 4x^3 + 29x^2 + 3x - 13$, $K = k(\sqrt{-2993\xi^5 + 7230\xi^4 + 18937\xi^3 - 38788\xi^2 - 32096\xi + 44590})$. Then K is totally real and has an infinite tamely ramified 2-tower $\mathcal{K}$, for which, under GRH, we have:*

$$\mathrm{BS}_{\mathrm{lower}} \leq \mathrm{BS}(\mathcal{K}) \leq \mathrm{BS}_{\mathrm{upper}},$$

where $\mathrm{BS}_{\mathrm{lower}} \approx 0.79144\ldots$, $\mathrm{BS}_{\mathrm{upper}} \approx 0.81209\ldots$.

However, unconditionally (without GRH), the estimates for totally complex fields that may be obtained using the methods developed by Tsfasman and Vlăduţ lead to slightly worse results, than those already known from [11]. This is due to a rather large number of prime ideals of small norm in the field K. For the same reasons the upper bounds for the Brauer-Siegel ratio for other fields constructed in [3] are too high, though the lower bounds are still good enough.

Finally we present the table (the ameliorated version of the table of [11]), where all the bounds and estimates are given together:

		lower bound	lower example	upper example	upper bound
	all fields	0.5165	0.5649-0.5975	1.0602-1.0798	1.0938
GRH	totally real	0.7419	0.7914-0.8121	1.0602-1.0798	1.0938
	totally complex	0.5165	0.5649-0.5975	1.0482-1.0653	1.0764
	all fields	0.4087	0.5939-0.6208	1.0602-1.1133	1.1588
Unconditional	totally real	0.6625	0.8009-0.9081	1.0602-1.1133	1.1588
	totally complex	0.4087	0.5939-0.6208	1.0482-1.1026	1.1310

2. Proof of Theorem 1

Let $\zeta_K(s)$ be the Dedekind zeta function of the number field K and $\varkappa_K$ its residue at $s = 1$. By w_K we denote the number of roots of unity in K, and by r_1, r_2 the number of real and complex places of K respectively. We have the following residue formula (see [4], Chapter VIII, Section 3):

$$\varkappa = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{D_K}}.$$

Since

$$\sqrt{w_K/2} \leq \varphi(w_K) = [\mathbb{Q}(\zeta_{w_K}) : \mathbb{Q}] \leq [K : \mathbb{Q}] = n_K,$$

we note that $w_K \leq 2n_K^2$ so $\log w_{K_j}/g_{K_j} \to 0$. Thus, it is enough to prove that $\log \varkappa_{K_j}/\log D_{K_j} \to 0$.

As for the upper bound we have

Theorem 5 (See [5], Theorem 1). *Let K be a number field of degree $n \geq 2$. Then,*

$$\varkappa_K \leq \left(\frac{e \log D_K}{2(n-1)}\right)^{n-1}. \tag{2}$$

Moreover, $1/2 \leq \rho < 1$ and $\zeta_K(\rho) = 0$ imply

$$\varkappa_K \leq (1-\rho)\left(\frac{e \log D_K}{2n}\right)^n. \tag{3}$$

Using the estimate (2) we get (even without the assumption of almost normality) the "easy inequality":

$$\frac{\log \varkappa_{K_j}}{\log D_{K_j}} \leq \frac{n_j - 1}{\log D_{K_j}}\left(\log \frac{e}{2} + \log \frac{\log D_{K_j}}{n_j - 1}\right) \to 0.$$

As for the lower bound the business is much more tricky and we will proceed to the proof after giving a few preliminary statements.

Let K be a number field other than $\mathbb{Q}$. A real number ρ is called an *exceptional zero* of $\zeta_K(s)$ if $\zeta_K(\rho) = 0$ and

$$1 - (4\log D_K)^{-1} \leq \rho < 1;$$

an exceptional zero ρ of $\zeta_K(s)$ is called its *Siegel zero* if

$$1 - (16\log D_K)^{-1} \leq \rho < 1.$$

Our proof will be based on the following fundamental property of Siegel zeroes proved by Stark:

Theorem 6 (See [10], Lemma 10). *Let K be an almost normal number field, and let ρ be a Siegel zero of $\zeta_K(s)$. Then there exists a quadratic subfield k of K such that $\zeta_k(\rho) = 0$.*

The next estimate is also due to Stark:

Theorem 7 (See [10], Lemma 4 or [6], Theorem 1). *Let K be a number field and let ρ be the exceptional zero of $\zeta_K(s)$ if it exists and $\rho = 1 - (4 \log D_K)^{-1}$ otherwise. Then there is an absolute constant $c < 1$ (effectively computable) such that*

$$\varkappa_K > c(1 - \rho) \qquad (4)$$

Our proof of Theorem 1 will be similar to the proof of the classical Brauer-Siegel theorem given in [7]. We will use the Brauer-Siegel result for quadratic fields, a simple proof of which is given in [2]. There are two cases to consider.

1. First, assume that $\zeta_{K_j}(s)$ has no Siegel zero. From (4) we deduce that

$$\varkappa_{K_j} > c(1 - \rho) \geq c\left(1 - \left(1 - \frac{1}{16 \log D_{K_j}}\right)\right) = \frac{c}{16 \log D_{K_j}}. \qquad (5)$$

2. Second, assume that there exists a Siegel zero ρ of $\zeta_{K_j}(s)$. From Theorem 6 we see that there exists a quadratic subfield k_j of K_j such that $\zeta_{k_j}(\rho) = 0$. Applying (3) and (4) we obtain:

$$\varkappa_{K_j} = \frac{\varkappa_{K_j}}{\varkappa_{k_j}} \varkappa_{k_j} \geq \frac{c(1 - \rho)}{(1 - \rho)\left(\frac{e \log D_{k_j}}{4}\right)^2} \varkappa_{k_j} = \frac{16c}{e^2 \log^2 D_{k_j}} \varkappa_{k_j}. \qquad (6)$$

If the number of fields K_j for which the second case holds is finite, then, using the fact that $\log D_{K_j} \to \infty$, we get the desired lower estimate from (5).

Otherwise, we note that for a number field there exists at most one exceptional zero (See [10], Lemma 3), so, applying this statement to the fields k_j, we get that only finitely many of them may be isomorphic to each other and so $D_{k_j} \to \infty$ as $j \to \infty$. Thus we may use the Brauer-Siegel result for quadratic fields:

$$\frac{\log \varkappa_{k_j}}{\log D_{K_j}} \leq \frac{\log \varkappa_{k_j}}{\log D_{k_j}} \to 0.$$

Finally from (6), we get:

$$\frac{\log \varkappa_{K_j}}{\log D_{K_j}} \geq \frac{16c}{e^2 \log D_{K_j}} - 2\frac{\log \log D_{k_j}}{\log D_{K_j}} + \frac{\log \varkappa_{k_j}}{\log D_{K_j}} \to 0.$$

This concludes the proof. $\qquad \square$

Remark 8. Our proof of Theorem 1 is explicit and effective if all the fields in the family $\mathcal{K}$ contain no quadratic subfield and thus the corresponding zeta function does not have Siegel zeroes.

3. Proof of Theorem 4

First we recall briefly some constructions related to class field towers. Let us fix a prime number ℓ. For a finitely generated pro-ℓ group G, we let $d(G) = \dim_{\mathbb{F}_\ell} H^1(G, \mathbb{F}_\ell)$ be its generator rank. Let T be a finite set of ideals of a number field K such that no prime in T is a divisor of ℓ. We denote by K_T the maximal ℓ-extension of K unramified outside T, $G_T = \mathrm{Gal}(K_T/K)$. We let

$$\theta_{K,T} = \begin{cases} 1, & \text{if } T \neq \varnothing \text{ and } K \text{ contains a primitive } \ell\text{th root of unity;} \\ 0, & \text{otherwise.} \end{cases}$$

Then we have (see [9], theorems 1 and 5):

Theorem 9. *If $d(G_T) \geq 2 + 2\sqrt{r_1(K) + r_2(K) + \theta_{K,T}}$, then K_T is infinite.*

To estimate $d(G_T)$ we use the following theorem

Theorem 10 (See [8], section 2). *Let K/k be a finite Galois extension, $r_1 = r_1(k)$, $r_2 = r_2(k)$, ρ be the number of real places of k, ramified in K, t be the number of primes in k, ramified in K. We set $\delta_\ell = 1$, if k contains a primitive root of degree ℓ of unity and $\delta_\ell = 0$ otherwise. Then we have:*

$$d(G_T) \geq d(G_\varnothing) \geq t - r_1 - r_2 + \rho - \delta_\ell$$

The number field arithmetic behind the construction of our theorem 4 was mainly carried out with the help of the computer package PARI. However, we would like to present our examples in the way suitable for non-computer check. We give here the proof of the first part of our theorem, as the proof of the second part is very much similar and may be carried out simply by repeating all the steps of the proof given here.

The following construction is taken from [3]. We let $k = \mathbb{Q}(\xi)$, where ξ is a root of $f(x) = x^6 + x^4 - 4x^3 - 7x^2 - x + 1$. Then k is a field of signature $(4, 1)$ and discriminant $d_f = d_k = -23 \cdot 35509$. Its ring of integers is $\mathcal{O}_k = \mathbb{Z}[\xi]$ and its class number is equal to 1. The principal ideal of norm $7 \cdot 13 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31$ generated by $\eta = -671\xi^5 + 467\xi^4 - 994\xi^3 + 3360\xi^2 + 2314\xi - 961$ factors into eight different prime ideals of $\mathcal{O}_k$. In fact, one may see that $\eta = \pi_7 \pi_{13} \pi_{19} \pi'_{19} \pi_{23} \pi'_{23} \pi_{29} \pi_{31}$, where

$$\begin{aligned}
\pi_7 &= -9\xi^5 + 6\xi^4 - 13\xi^3 + 44\xi^2 + 31\xi - 12, \\
\pi_{13} &= -7\xi^5 + 5\xi^4 - 11\xi^3 + 36\xi^2 + 23\xi - 9, \\
\pi_{19} &= 5\xi^5 - 4\xi^4 + 8\xi^3 - 26\xi^2 - 15\xi + 6, \\
\pi'_{19} &= 5\xi^5 - 3\xi^4 + 7\xi^3 - 24\xi^2 - 20\xi + 6, \\
\pi_{23} &= -5\xi^5 + 4\xi^4 - 8\xi^3 + 26\xi^2 + 15\xi - 9, \\
\pi'_{23} &= 6\xi^5 - 4\xi^4 + 9\xi^3 - 30\xi^2 - 22\xi + 6, \\
\pi_{29} &= 11\xi^5 - 8\xi^4 + 17\xi^3 - 56\xi^2 - 35\xi + 16, \\
\pi_{31} &= 7\xi^5 - 5\xi^4 + 11\xi^3 - 36\xi^2 - 22\xi + 7.
\end{aligned}$$

$K = k(\sqrt{\eta})$ is a totally complex field of degree 12 over $\mathbb{Q}$ with the relative discriminant $\mathcal{D}_{K/k}$ equal to (η) as $\eta = \beta^2 + 4\gamma$, where $\beta = \xi^5 + \xi^4 + \xi^3 + 1$, $\gamma = -173\xi^5 + 112\xi^4 - 270\xi^3 + 815\xi^2 + 576\xi - 237$. From this we see that $d_K = 7 \cdot 13 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31 \cdot 23^2 \cdot 35509^2$. From Theorem 10 we deduce that

$$d(G_\varnothing) \geq t - r_1(k) - r_2(k) + \rho - 1 = 8 - 4 - 1 + 4 - 1 = 6.$$

The right hand side of the inequality from Theorem 9 is equal to $2 + 2\sqrt{6} \approx 6.8989 < 7$, so it is enough to show that $d(G_T) > d(G_\varnothing)$, and to do this it is enough to construct a set of prime ideals T and an extension of K, ramified exactly at T.

Let $\pi_3 = -6\xi^5 + 4\xi^4 - 9\xi^3 + 30\xi^2 + 21\xi - 7$ be the generator of a prime ideal of norm 3 in $\mathcal{O}_k$ and T be the set consisting of one prime ideal of $\mathcal{O}_K$ over $\pi_3\mathcal{O}_k$. We see that $\pi_3\pi_{19} = 11\xi^5 - 8\xi^4 + 17\xi^3 - 56\xi^2 - 35\xi + 14 = \rho^2 + 4\sigma$, where $\rho = \xi^5 + \xi^3 + \xi^2 + 1$, $\sigma = 2\xi^5 - 8\xi^4 - 14\xi^3 - 28\xi^2 - 9\xi + 5$, so $k(\sqrt{\pi_3\pi_{19}})/k$ is ramified exactly at π_3 and π_{19}. But π_{19} already ramifies in K that is why $K(\sqrt{\pi_3\pi_{19}})/K$ is ramified exactly at T. Thus we have showed that $d(G_T) \geq 7$ and K_T/K is indeed infinite.

To complete our proof we need a few more results.

Theorem 11 (GRH Basic Inequality, see [11], Theorem 3.1). *For an asymptotitically exact family of number fields under GRH one has:*

$$\sum_q \frac{\phi_q \log q}{\sqrt{q} - 1} + \phi_\mathbb{R}\left(\log 2\sqrt{2\pi} + \frac{\pi}{4} + \frac{\gamma}{2}\right) + \phi_\mathbb{C}(\log 8\pi + \gamma) \leq 1, \qquad (7)$$

the sum beeing taken over all prime powers q.

Theorem 12 (See [3], Theorem 1). *Let K be a number field of degree n over $\mathbb{Q}$, such that K_T is infinite and assume that $K_T = \bigcup_{i=1}^\infty K_i$. Then*

$$\lim_{i \to \infty} \frac{g_i}{n_i} \leq \frac{g_K}{n_K} + \frac{\sum_{\mathfrak{p} \in T} \log(N_{K/\mathbb{Q}}\mathfrak{p})}{2n_K}.$$

For our previously constructed field K the genus is equal to $g_K \approx 25.3490\ldots$. From Theorem 12 we easily see that $\phi_\mathbb{R} = 0$ and $\frac{12}{2g_K + 2\log 3} \leq \phi_\mathbb{C} \leq \frac{12}{2g_K}$, i. e., $0.23669 < \phi_\mathbb{C} < 0.22687$. The lower bound for $\mathrm{BS}(K_T)$ is clearly equal to

$$\mathrm{BS}_{\mathrm{lower}} = 1 - \phi_\mathbb{R} \log 2 - \phi_\mathbb{C} \log(2\pi) \leq 0.56498\ldots.$$

Knowing the decomposition in K of small primes of $\mathbb{Q}$, we may now apply the linear programming approach to get the upper bound for $\mathrm{BS}(K_T)$. This is done using the explicit formula (1) for the Brauer-Siegel ratio along with the basic inequality (7) and the inequality

$$\sum_{m=1}^\infty m\phi_{p^m} \leq \phi_\mathbb{R} + 2\phi_\mathbb{C},$$

taken as the restrictions. This was done using the PARI package. As the calculations are rather cumbersome we will give here only the final result: $\mathrm{BS}_{\mathrm{upper}} \approx 0.59748\ldots$, and the bound is attained for $\phi_7 = \phi_9 = \phi_{13} = 0.03944\ldots, \phi_{19} = 0.01002\ldots$. $\square$

Acknowledgements

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