Jacobians among Abelian threefolds: a formula of Klein and a question of Serre

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Abstract

Let k be a field and f be a Siegel modular form of weight $h \geqslant 0$ and genus $g > 1$ over k. Using f, we define an invariant of the k-isomorphism class of a principally polarized abelian variety $(A, a)/k$ of dimension g. Moreover when (A, a) is the Jacobian of a smooth plane curve, we show how to associate to f a classical plane invariant. As straightforward consequences of these constructions when $g = 3$ and $k \subset \mathbb{C}$ we obtain (i) a new proof of a formula of Klein linking the modular form χ_{18} to the square of the discriminant of plane quartics ; (ii) a proof that one can decide when (A, a) is a Jacobian over k by looking whether the value of χ_{18} at (A, a) is a square in k. This answers a question of J.-P. Serre. Finally, we study the possible generalizations of this approach for $g > 3$.

1. Introduction

1.1 Torelli theorem

Let k be an algebraically closed field and $g \geqslant 1$ be an integer. If X is a (nonsingular irreducible projective) curve of genus g over k, Torelli's theorem states that the map $X \mapsto (JacX, j)$, associating to X its Jacobian together with the canonical polarization j, is injective. The determination of the image of this map is a long time studied question.

When $g = 3$, the moduli space A_g of principally polarized abelian varieties of dimension g and the moduli space M_g of nonsingular algebraic curves of genus g are both of dimension $g(g+1)/2 = 3g - 3 = 6$. According to Hoyt [12] and Oort and Ueno [25], the image of M_3 is exactly the space of indecomposable principally polarized abelian threefolds. Moreover if $k = \mathbb{C}$, Igusa [17] characterized the locus of decomposable abelian threefolds and that of hyperelliptic Jacobians, making use of two particular modular forms Σ_{140} and χ_{18} on the Siegel upper half space of degree 3.

Assume now that k is any field and $g \geqslant 1$. J.-P. Serre noticed in [22] that a precise form of Torelli's theorem reveals a mysterious obstruction for a geometric Jacobian to be a Jacobian over k. More precisely, he proved the following:

Theorem 1.1.1. *Let (A, a) be a principally polarized abelian variety of dimension $g \geqslant 1$ over k, and assume that (A, a) is isomorphic over $\overline{k}$ to the Jacobian of a curve X_0 of genus g defined over $\overline{k}$. The following alternative holds :*

(i) *If X_0 is hyperelliptic, there is a curve X/k isomorphic to X_0 over $\overline{k}$ such that (A, a) is k-isomorphic to $(JacX, j)$.*

(ii) *If X_0 is not hyperelliptic, there is a curve X/k isomorphic to X_0 over $\overline{k}$, and a quadratic character*

$$\varepsilon : Gal(k^{sep}/k) \longrightarrow \{\pm 1\}$$

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such that the twisted abelian variety $(A, a)_\varepsilon$ is k-isomorphic to $(JacX, j)$. The character ε is trivial if and only if (A, a) is k-isomorphic to a Jacobian.

Thus, only case (i) occurs if $g = 1$ or $g = 2$, with all curves being elliptic or hyperelliptic.

1.2 Curves of genus 3

Assume now again $g = 3$. Let there be given an indecomposable principally polarized abelian threefold (A, a) defined over k. In a letter to J. Top [28], J.-P. Serre asked a twofold question:

— *How to decide, knowing only (A, a), that X is hyperelliptic ?*

— *If X is not hyperelliptic, how to compute the quadratic character ε ?*

Assume that $k \subset \mathbb{C}$. The first question can easily be answered using the forms Σ_{140} and χ_{18}. As for the second question, roughly speaking, Serre suggested that ε is trivial if and only if χ_{18} is a square in $k^\times$ (see Th.4.1.2 for a more precise formulation). This assertion was motivated by a formula of Klein [20] relating the modular form χ_{18} (in the notation of Igusa) to the square of the discriminant of plane quartics, which more or less gives the 'only if' part of the claim. In [21], two of the authors justified Serre's assertion for a three dimensional family of abelian varieties and in particular determined the absolute constant involved in Klein's formula.

In this article we prove that Serre's assertion is valid for any abelian threefold. In order to do so, we start by taking a broader point of view, valid for any $g > 1$.

(i) We look at the action of $\overline{k}$-isomorphisms on Siegel modular forms defined over k and we define invariants of k-isomorphism classes of abelian varieties over k.

(ii) We link Siegel modular forms, Teichmüller modular forms and invariants of plane curves.

Once these two goals are achieved, Serre's assertion can be rephrased as the following strategy

— use (ii) to find a Siegel modular form whose 'values' are a suitable power in k on the Jacobian locus;

— use (i) to distinguish between Jacobians and their twists.

For $g = 3$, Klein's formula shows that the form χ_{18} is a square on the Jacobian locus and that this is enough to characterize this locus. On the other hand, we show that this is no longer the case for the natural generalization χ_h, $h = 2^{g-2}(2^g + 1)$, when $g > 3$.
The relevance of Klein's formula in this problem was one of Serre's insights. We would like to point out that we do not actually need the full strength of Klein's formula to work out our strategy. Indeed, we do not go all the way from Siegel modular form to invariants. We use instead a formula due to Ichikawa relating χ_{18} to the square of a Teichmüller modular form (see Rem.4.1.3). However we think that the connection between Siegel modular forms and invariants is interesting enough in its own, besides the fact that it gives a new proof of Klein's formula.

The paper is organized as follows. In §2, we review the necessary elements from the theory of Siegel and Teichmüller modular forms. Only §2.4 is original: we introduce the action of isomorphisms and see how the action of twists is reflected on the values of modular forms. In §3, we link modular forms and certain invariants of ternary forms. Finally in §4 we deal with the case $g = 3$. We give first a proof of Klein's formula and then we justify the validity of Serre's assertion. Finally we explain the reasons behind the failure of the obvious generalization of the theory in higher dimensions and state some natural questions.

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2. Siegel and Teichmüller modular forms

2.1 Geometric Siegel modular forms

The references are [4], [5], [7], [10]. Let $g > 1$ and $n > 0$ be two integers and $\mathsf{A}_{g,n}$ be the moduli stack of principally polarized abelian schemes of relative dimension g with symplectic level n structure. Let $\pi : \mathsf{V}_{g,n} \longrightarrow \mathsf{A}_{g,n}$ be the universal abelian scheme, fitted with the zero section $\varepsilon : \mathsf{A}_{g,n} \longrightarrow \mathsf{V}_{g,n}$, and

$$\pi_*\Omega^1_{\mathsf{V}_{g,n}/\mathsf{A}_{g,n}} = \varepsilon^*\Omega^1_{\mathsf{V}_{g,n}/\mathsf{A}_{g,n}} \longrightarrow \mathsf{A}_{g,n}$$

the rank g bundle induced by the relative regular differential forms of degree one on $\mathsf{V}_{g,n}$ over $\mathsf{A}_{g,n}$. The relative canonical bundle over $\mathsf{A}_{g,n}$ is the line bundle

$$\underline{\omega} = \bigwedge^g \varepsilon^*\Omega^1_{\mathsf{V}_{g,n}/\mathsf{A}_{g,n}}.$$

For a projective nonsingular variety X defined over a field k, we denote by

$$\Omega^1_k[X] = H^0(X, \Omega^1_X \otimes k)$$

the finite dimensional k-vector space of regular differential forms on X defined over k. Hence, the fibre of the bundle $\Omega^1_{\mathsf{V}_{g,n}/\mathsf{A}_{g,n}}$ over $A \in \mathsf{A}_{g,n}(k)$ is equal to $\Omega^1_k[A]$, and the fibre of $\underline{\omega}$ is the one-dimensional vector space

$$\underline{\omega}[A] = \bigwedge^g \Omega^1_k[A].$$

We put $\mathsf{A}_g = \mathsf{A}_{g,1}$ and $\mathsf{V}_g = \mathsf{V}_{g,1}$. Let R be a commutative ring and h a positive integer. A *geometric Siegel modular form* of genus g and weight h over R is an element of the R-module

$$\mathbf{S}_{g,h}(R) = \Gamma(\mathsf{A}_g \otimes R, \underline{\omega}^{\otimes h}).$$

Note that for any $n \geqslant 1$, we have an isomorphism

$$\mathsf{A}_g \simeq \mathsf{A}_{g,n}/Sp_{2g}(\mathbb{Z}/n\mathbb{Z}).$$

If $n \geqslant 3$, as shown in [24], from the rigidity lemma of Serre [27] we can deduce that the moduli space $\mathsf{A}_{g,n}$ can be represented by a smooth scheme over $\mathbb{Z}[\zeta_n, 1/n]$. Hence, for any algebra R over $\mathbb{Z}[\zeta_n, 1/n]$, the module $\mathbf{S}_{g,h}(R)$ is the submodule of

$$\Gamma(\mathsf{A}_{g,n} \otimes_{\mathbb{Z}[\zeta_n,1/n]} R, \underline{\omega}^{\otimes h})$$

consisting of the elements invariant under $Sp_{2g}(\mathbb{Z}/n\mathbb{Z})$.
Assume now that $R = k$ is a field. If $f \in \mathbf{S}_{g,h}(k)$, A is a p.p.a.v. of dimension g defined over k and ω is a basis of $\underline{\omega}_k[A]$, define

$$f(A, \alpha) = f(A)/\omega^{\otimes h}. \tag{1}$$

In this way such a modular form defines a rule which assigns the element $f(A, \omega) \in k$ to every such pair (A, ω) and such that:

(i) $f(A, \lambda\omega) = \lambda^{-h} f(A, \omega)$ for any $\lambda \in k^\times$.

(ii) $f(A, \omega)$ depends only on the $\overline{k}$-isomorphism class of the pair (A, ω).

Conversely, such a rule defines a unique $f \in \mathbf{S}_{g,h}(k)$. This definition is a straightforward generalization of that of Deligne-Serre [6] and Katz [19] if $g = 1$.

2.2 Complex uniformisation

Assume $R = \mathbb{C}$. Let

$$\mathbb{H}_g = \left\{\tau \in \mathbf{M}_g(\mathbb{C}) \mid {}^t\tau = \tau,\ \operatorname{Im}\tau > 0\right\}$$

be the Siegel upper half space of genus g and $\Gamma = Sp_{2g}(\mathbb{Z})$. As explained in [4, §2], The complex orbifold $\mathsf{A}_g(\mathbb{C})$ can be expressed as the quotient $\Gamma\backslash\mathbb{H}_g$ where Γ acts by

$$M.\tau = (a\tau + b)\cdot(c\tau + d)^{-1} \quad \text{if} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

The group $\mathbb{Z}^{2g}$ acts on $\mathbb{H}_g \times \mathbb{C}^g$ by

$$v.(\tau, z) = (\tau, z + \tau m + n) \quad \text{if} \quad v = \begin{pmatrix} m \\ n \end{pmatrix} \in \mathbb{Z}^{2g}.$$

If $\mathbb{U}_g = \mathbb{Z}^{2g}\backslash(\mathbb{H}_g \times \mathbb{C}^g)$, the projection

$$\pi : \mathbb{U}_g \longrightarrow \mathbb{H}_g$$

defines a universal principally polarized abelian variety with fibres

$$A_\tau = \pi^{-1}(\tau) = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g).$$

Let $j(M,\tau) = c\tau + d$ and define the action of Γ on $\mathbb{H}_g \times \mathbb{C}^g$ by

$$M.(\tau, (z_1, \ldots, z_g)) = (M.\tau, {}^tj(M,\tau)^{-1}\cdot(z_1, \ldots, z_g)) \quad \text{if } M \in \Gamma.$$

The map ${}^tj(M,\tau)^{-1} : \mathbb{C}^g \to \mathbb{C}^g$ induces an isomorphism:

$$\varphi_M : A_\tau \longrightarrow A_{M.\tau}.$$

Hence, $\mathsf{V}_g(\mathbb{C}) \simeq \Gamma\backslash\mathbb{U}_g$ and the following diagram is commutative:

$$\begin{tikzcd} \Gamma\backslash\mathbb{U}_g \arrow[r, "\sim"] \arrow[d, "\pi"'] & \mathsf{V}_g(\mathbb{C}) \arrow[d, "\pi"'] \\ \Gamma\backslash\mathbb{H}_g \arrow[r, "\sim"] & \mathsf{A}_g(\mathbb{C}) \end{tikzcd}$$

As in [7, p. 141], let

$$\zeta = \frac{dq_1}{q_1}\wedge\ldots\wedge\frac{dq_g}{q_g} = (2i\pi)^g dz_1\wedge\cdots\wedge dz_g \in \Gamma(\mathbb{H}_g, \underline{\omega})$$

with $(z_i, \ldots, z_g) \in \mathbb{C}^g$ and $(q_i, \ldots, q_g) = (e^{2i\pi z_1}, \ldots e^{2i\pi z_g})$. This section of the canonical bundle is a basis of $\underline{\omega}[A_\tau]$ for all $\tau \in \mathbb{H}_g$ and the relative canonical bundle of $\mathbb{U}_g/\mathbb{H}_g$ is trivialized by ζ :

$$\underline{\omega}_{\mathbb{U}_g/\mathbb{H}_g} = \bigwedge^g \Omega^1_{\mathbb{U}_g/\mathbb{H}_g} \simeq \mathbb{H}_g \times \mathbb{C}\cdot\zeta.$$

The group Γ acts on $\underline{\omega}_{\mathbb{U}_g/\mathbb{H}_g}$ by

$$M.(\tau, \zeta) = (M.\tau, \det j(M,\tau)\cdot\zeta) \quad \text{if } M \in \Gamma,$$

in such a way that

$$\varphi_M^*(\zeta_{M.\tau}) = \det j(M,\tau)^{-1}\zeta_\tau.$$

Thus, a geometric Siegel modular form f of weight h becomes an expression

$$f(A_\tau) = \widetilde{f}(\tau)\cdot\zeta^{\otimes h},$$

where $\widetilde{f}$ belongs to the well-known vector space $\mathbf{R}_{g,h}(\mathbb{C})$ of *analytic Siegel modular forms* of weight h on $\mathbb{H}_g$, consisting of complex holomorphic functions $\phi(\tau)$ on $\mathbb{H}_g$ satisfying

$$\phi(M.\tau) = \det j(M.\tau)^h \phi(\tau)$$

for any $M \in Sp_{2g}(\mathbb{Z})$. Note that by Koecher principle [10, p. 11], the condition of holomorphy at ∞ is automatically satisfied since $g > 1$. The converse is also true [7, p. 141]:

Proposition 2.2.1. *If $f \in \mathbf{S}_{g,h}(\mathbb{C})$ and $\tau \in \mathbb{H}_g$, let*

$$\widetilde{f}(\tau) = f(A_\tau)/\zeta^{\otimes h} = (2i\pi)^{-gh} f(A_\tau)/(dz_1 \wedge \cdots \wedge dz_g)^{\otimes h}.$$

Then the map $f \mapsto \widetilde{f}$ is an isomorphism $\mathbf{S}_{g,h}(\mathbb{C}) \xrightarrow{\sim} \mathbf{R}_{g,h}(\mathbb{C})$. □

2.3 Teichmüller modular forms

Let $g > 1$ and $n > 0$ be positive integers and let $\mathsf{M}_{g,n}$ denote the moduli stack of smooth and proper curves of genus g with symplectic level n structure [5]. Let $\pi : \mathsf{C}_{g,n} \longrightarrow \mathsf{M}_{g,n}$ be the universal curve, and let $\underline{\lambda}$ be the invertible sheaf associated to the *Hodge bundle*, namely

$$\underline{\lambda} = \bigwedge^{g} \pi_* \Omega^1_{\mathsf{C}_{g,n}/\mathsf{M}_{g,n}}.$$

For an algebraically closed field k the fibre over $C \in \mathsf{M}_{g,n}(k)$ is the one dimensional vector space $\underline{\lambda}[C] = \bigwedge^g \Omega^1_k[C]$.

Let R be a commutative ring and h a positive integer. A *Teichmüller modular form* of genus g and weight h over R is an element of

$$\mathbf{T}_{g,h}(R) = \Gamma(\mathsf{M}_g \otimes R, \underline{\lambda}^{\otimes h}).$$

These forms have been thoroughly studied by Ichikawa [13], [14], [15], [16]. As in the case of the moduli space of abelian varieties, for any $n \geqslant 1$ we have

$$\mathsf{M}_g \simeq \mathsf{M}_{g,n}/Sp_{2g}(\mathbb{Z}/n\mathbb{Z}),$$

and $\mathsf{M}_{g,n}$ can be represented by a smooth scheme over $\mathbb{Z}[\zeta_n, 1/n]$ if $n \geqslant 3$. Then, for any algebra R over $\mathbb{Z}[\zeta_n, 1/n]$, the module $\mathbf{T}_{g,h}(R)$ is the submodule of

$$\Gamma(\mathsf{M}_{g,n} \otimes_{\mathbb{Z}[\zeta_n,1/n]} R, \underline{\lambda}^{\otimes h})$$

invariant under $Sp_{2g}(\mathbb{Z}/n\mathbb{Z})$.

Let C/k be a genus g curve. Let $\lambda_1, \ldots, \lambda_g$ be a basis of $\Omega^1_k[C]$ and $\lambda = \lambda_1 \wedge \ldots \wedge \lambda_g$ a basis of $\underline{\lambda}[C]$. As for Siegel modular forms in (1), for a Teichmüller modular form $f \in \mathbf{T}_{g,h}(k)$ we define

$$f(C, \lambda) = f(C)/\lambda^{\otimes h} \in k.$$

Ichikawa asserts the following proposition:

Proposition 2.3.1. *The Torelli map $\theta : \mathsf{M}_g \longrightarrow \mathsf{A}_g$, associating to a curve C its Jacobian $JacC$ with the canonical polarization j, satisfies $\theta^*\underline{\omega} = \underline{\lambda}$, and induces for any commutative ring R a linear map*

$$\theta^* : \mathbf{S}_{g,h}(R) = \Gamma(\mathsf{A}_g \otimes R, \underline{\omega}^{\otimes h}) \longrightarrow \mathbf{T}_{g,h}(R) = \Gamma(\mathsf{M}_g \otimes R, \underline{\lambda}^{\otimes h}),$$

such that $[\theta^* f](C) = \theta^*[f(JacC)]$. Fixing a basis λ of $\underline{\lambda}[C]$, this is

$$f(JacC, \omega) = [\theta^* f](C, \lambda) \qquad \text{if } \theta^*\omega = \lambda.$$

□

2.4 Action of isomorphisms

Suppose $\phi : (A', a') \longrightarrow (A, a)$ is a $\overline{k}$-isomorphism of principally polarized abelian varieties. Let $\omega_1, \ldots, \omega_g \in \Omega^1{}_{\overline{k}}[A]$ and $\omega = \omega_1 \wedge \ldots \wedge \omega_g \in \underline{\omega}[A]$. Then by definition

$$f(A, \omega) = f(A', \gamma)$$

where $\gamma_i = \phi^*(\omega_i)$ is a basis of $\Omega^1{}_{\overline{k}}[A']$ and $\gamma = \gamma_1 \wedge \ldots \wedge \gamma_g \in \underline{\omega}[A']$. If $\omega'_1, \ldots, \omega'_g$ is another basis of $\Omega^1{}_{\overline{k}}[A']$ and $\omega' = \omega'_1 \wedge \ldots \wedge \omega'_g$, we denote by $M_\phi \in GL_g(\overline{k})$ the matrix of the basis (γ_i) in the basis (ω'_i). We can easily see that

Proposition 2.4.1. *In the above notation,*

$$f(A, \omega) = \det(M_\phi)^h \cdot f(A', \omega'). \quad \square$$

First of all, from this formula applied to the action of -1, we deduce that, if k is a field of characteristic different from 2, then $\mathbf{S}_{g,h}(k) = \{0\}$ if gh is odd. From now on we assume that gh is even and $char k \neq 2$.

Corollary 2.4.2. *Let (A, a) be a principally polarized abelian variety of dimension g defined over k and $f \in \mathbf{S}_{g,h}(k)$. Let $\omega_1, \ldots, \omega_g$ be a basis of $\Omega^1_k[A]$, and let $\omega = \omega_1 \wedge \ldots \wedge \omega_g \in \underline{\omega}[A]$. Then the quantity*

$$\bar{f}(A) = f(A, \omega) \bmod^\times k^{\times h} \in k/k^{\times h}$$

does not depend on the choice of the basis of $\Omega^1_k[A]$. In particular $\bar{f}(A)$ is an invariant of the k-isomorphism class of A. $\square$

Corollary 2.4.3. *Assume that g is odd. Let $f \in \mathbf{S}_{g,h}(k)$ and $\phi : A' \longrightarrow A$ a non trivial quadratic twist. There exists $c \in k \setminus k^2$ such that $\bar{f}(A) = c^{h/2} \bar{f}(A')$. Thus, if $\bar{f}(A) \neq 0$ then $\bar{f}(A)$ and $\bar{f}(A')$ do not belong to the same class in $k^\times / k^{\times h}$.*

Proof. Assume that ϕ is given by the quadratic character ε of $Gal(\overline{k}/k)$. Then

$$d^\sigma = \varepsilon(\sigma)^g \cdot d, \text{ where } \quad d = \det(M_\phi) \in \overline{k}, \quad \sigma \in Gal(\overline{k}/k).$$

Assume that g is odd. Then by our assumption h is even, and $d^2 = \varepsilon(\sigma)^g d d^\sigma \in k$. But $d \notin k$ since there exists σ such that $\varepsilon(\sigma) = -1$. Using Prop.2.4.1 we find that

$$f(A, \omega) = (d^2)^{h/2} f(A', \omega').$$

Since d^2 is not a square in k, if $\bar{f}(A) \neq 0$ then $\bar{f}(A)$ and $\bar{f}(A')$ belong to two different classes. $\square$

Let now (A, a) be a principally polarized abelian variety of dimension g defined over $\mathbb{C}$. Let $\omega_1, \ldots, \omega_g$ be a basis of $\Omega^1_{\mathbb{C}}[A]$ and $\omega = \omega_1 \wedge \ldots \wedge \omega_g \in \underline{\omega}[A]$. Let $\gamma_1, \ldots \gamma_{2g}$ be a symplectic basis (for the polarization a). The period matrix

$$\Omega = [\Omega_1 \ \Omega_2] = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_{2g}} \omega_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \omega_g & \cdots & \int_{\gamma_{2g}} \omega_g \end{pmatrix}$$

belongs to the set $\mathcal{R}_g \subset \mathbf{M}_{g,2g}(\mathbb{C})$ of Riemann matrices, and $\tau = \Omega_2^{-1} \Omega_1 \in \mathbb{H}_g$.

Proposition 2.4.4. *In the above notation,*

$$f(A, \omega) = (2i\pi)^{gh} \frac{\widetilde{f}(\tau)}{\det \Omega_2^h}.$$

Proof. The abelian variety A is isomorphic to $A_\Omega = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$ and $\omega \in \underline{\omega}[A]$ maps to $\xi = dz_1 \wedge \cdots \wedge dz_g \in \underline{\omega}[A_\Omega]$ under this isomorphism. The linear map $z \mapsto \Omega_2^{-1} z = z'$ induces the isomorphism

$$\varphi : A_\Omega \longrightarrow A_\tau = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g).$$

Let us denote $\xi' = dz'_1 \wedge \cdots \wedge dz'_g = (2i\pi)^{-g}\zeta$ in $\underline{\omega}[A_\tau]$. Thus, using Prop.2.4.1, Equation (1) and Prop.2.2.1, we obtain

$$\begin{aligned} f(A,\omega) &= f(A_\Omega, \xi) = \det \Omega_2^{-h} f(A_\tau, \xi') \\ &= \det \Omega_2^{-h} f(A_\tau)/\xi'^{\otimes h} = (2i\pi)^{gh} \det \Omega_2^{-h} f(\tau)/\zeta^{\otimes h} = (2i\pi)^{gh} \frac{\widetilde{f}(\tau)}{\det \Omega_2^h}, \end{aligned}$$

from which the proposition follows. $\square$

3. Invariants and modular forms

In this section k is an algebraically closed field of characteristic different from 2.

3.1 Invariants

We review some classical invariant theory. Let E be a vector space of dimension n over k. The left regular representation r of $GL(E)$ on the vector space $\mathsf{X}_d = \mathrm{Sym}^d(E^*)$ of homogeneous polynomials of degree d on E is given by

$$r(u) : F(x) \mapsto (u \cdot F)(x) = F(ux)$$

for $u \in GL(E)$, $F \in \mathsf{X}_d$ and $x \in E$. If U is an open subset of X_d stable under r, we still denote by r the left regular representation of $GL(E)$ on the k-algebra $\mathcal{O}(U)$ of regular functions on U, in such a way that

$$r(u) : \Phi(F) \mapsto (u \cdot \Phi)(F) = \Phi(u \cdot F),$$

if $u \in GL(E)$, $\Phi \in \mathcal{O}(U)$ and $F \in U$. If $h \in \mathbb{Z}$, we denote by $\mathcal{O}_h(U)$ the subspace of homogeneous elements of degree h, satisfying $\Phi(\lambda F) = \lambda^h \Phi(F)$ for $\lambda \in k^\times$ and $F \in U$. The subspaces $\mathcal{O}_h(U)$ are stable under r. An element $\Phi \in \mathcal{O}_h(U)$ is an *invariant of degree h on U* if

$$u \cdot \Phi = \Phi \quad \text{for every } u \in SL(E),$$

and we denote by $\mathsf{Inv}_h(U)$ the subspace of $\mathcal{O}_h(U)$ of invariants of degree h on U. If $\mathsf{Inv}_h(U) \neq \{0\}$, then $hd \equiv 0 \pmod{n}$, since the group μ_n of n-th roots of unity is in the kernel of r. Hence, if $\Phi \in \mathcal{O}(U)$, and if w and n are two integers such that $hd = nw$, the following statements are equivalent:

(i) $\Phi \in \mathsf{Inv}_h(U)$;

(ii) $u \cdot \Phi = (\det u)^w \Phi \quad$ for every $u \in GL(E)$.

If these conditions are satisfied, we call w the *weight* of Φ.

The *multivariate resultant* $Res(f_1, \ldots, f_n)$ of n forms $f_1, \ldots f_n$ in n variables with coefficients in k is an irreducible polynomial in the coefficients of $f_1, \ldots f_n$ which vanishes whenever $f_1, \ldots f_n$ have a common non-zero root. One requires that the resultant is irreducible over $\mathbb{Z}$, *i. e.* it has coefficients in $\mathbb{Z}$ with greatest common divisor equal to 1, and moreover

$$Res(x_1^{d_1}, \ldots, x_n^{d_n}) = 1$$

for any $(d_1, \ldots, d_n) \in \mathbb{N}^n$. The resultant exists and is unique. Now, let $F \in \mathsf{X}_d$, and denote $q_1, \ldots, q_n$ the partial derivatives of F. The *discriminant* of F is

$$Disc F = c_{n,d}^{-1} Res(q_1, \ldots, q_n), \quad \text{with} \quad c_{n,d} = d^{((d-1)^n - (-1)^n)/d},$$

the coefficient $c_{n,d}$ being chosen according to [28]. Hence, the projective hypersurface which is the zero locus of $F \in \mathsf{X}_d$ is nonsingular if and only if $DiscF \neq 0$. The discriminant is an irreducible polynomial in the coefficients of F, see for instance [8, Chap. 9, Ex. 1.6(a)]. From now on we restrict ourselves to the case $n = 3$, *i. e.* we consider invariants of ternary forms of degree d, and summarize the results that we shall need.

Proposition 3.1.1. *If $F \in \mathsf{X}_d$ is a ternary form, the discriminant*

$$DiscF = d^{-(d-1)(d-2)-1} \cdot Res(q_1, q_2, q_3)$$

where q_1, q_2, q_3 are the partial derivatives of F, is given by an irreducible polynomial over $\mathbb{Z}$ in the coefficients of F, and vanishes if and only if the plane curve C_F defined by F is singular. The discriminant is an invariant of X_d of degree $3(d-1)^2$ and weight $d(d-1)^2$. □

We refer to [8, p. 118] and [21] for an explicit formula for the discriminant, found by Sylvester.

Example 3.1.2 Ciani quartics. We recall some results whose proofs are given in [21]. Let

$$m = \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix} \in \mathrm{Sym}_3(k),$$

and for $1 \leqslant i \leqslant 3$, let $c_i = a_j a_k - b_i^2$ be the cofactor of a_i. If

$$\det(m) \neq 0, \ a_1 a_2 a_3 \neq 0 \text{ and } c_1 c_2 c_3 \neq 0$$

then

$$F_m(x, y, z) = a_1 x^4 + a_2 y^4 + a_3 z^4 + 2(b_1 y^2 z^2 + b_2 x^2 z^2 + b_3 x^2 y^2)$$

defines a non singular plane quartic. Moreover

$$DiscF_m = 2^{40}\, a_1\, a_2\, a_3\, (c_1\, c_2\, c_3)^2\, \det(m)^4.$$

Note that the discrepancy between the powers of 2 here and in [21, Prop.2.2.1] comes from the normalization by $c_{n,d}$.

3.2 Geometric invariants for nonsingular plane curves

Let E be a vector space of dimension 3 over k and $G = GL(E)$. The *universal curve* over the affine space $\mathsf{X}_d = \mathrm{Sym}^d(E)$ is the variety

$$\mathsf{Y}_d = \left\{(F, x) \in \mathsf{X}_d \times \mathbb{P}^2 \mid F(x) = 0\right\}.$$

The *nonsingular locus* of X_d is the principal open set

$$\mathsf{X}_d^0 = (\mathsf{X}_d)_{Disc} = \{F \in \mathsf{X}_d \mid Disc(F) \neq 0\}.$$

If Y_d^0 is the universal curve over the nonsingular locus X_d^0, the projection is a smooth surjective k-morphism

$$\pi : \mathsf{Y}_d^0 \longrightarrow \mathsf{X}_d^0$$

whose fibre over F is the non singular plane curve C_F.

We recall the classical way to write down an explicit k-basis of $\Omega^1[C_F] = H^0(C_F, \Omega^1)$ for $F \in \mathsf{X}_d^0(k)$ (see [3, p. 630]). Let

$$\eta_1 = \frac{f(x_2 dx_3 - x_3 dx_2)}{q_1}, \quad \eta_2 = \frac{f(x_3 dx_1 - x_1 dx_3)}{q_2}, \quad \eta_3 = \frac{f(x_1 dx_2 - x_2 dx_1)}{q_3},$$

where q_1, q_2, q_3 are the partial derivatives of F, and where f belongs to the space X_{d-3} of ternary forms of degree $d-3$. The forms η_i glue together and define a regular differential form $\eta_f(F) \in$

$\Omega^1[C_F]$. Since $\dim \mathsf{X}_{d-3} = (d-1)(d-2)/2 = g$, the linear map $f \mapsto \eta_f(F)$ defines an isomorphism

$$\mathsf{X}_{d-3} \xrightarrow{\sim} \Omega^1[C_F].$$

This implies that the sections $\eta_f \in \Gamma(\mathsf{X}_d^0, \pi_*\Omega^1_{\mathsf{Y}_d^0/\mathsf{X}_d^0})$ lead to a trivialization

$$\mathsf{X}_d^0 \times \mathsf{X}_{d-3} \xrightarrow{\sim} \pi_*\Omega^1_{\mathsf{Y}_d^0/\mathsf{X}_d^0}.$$

We denote $\eta_1, \ldots, \eta_g$ the sequence of sections obtained by substituting for f in η_f the g members of the canonical basis of X_{d-3}, enumerated according to the lexicographic order. Then

$$\eta = \eta_1 \wedge \ldots \wedge \eta_g$$

is a section of

$$\underline{\alpha} = \bigwedge^g \pi_*\Omega^1_{\mathsf{Y}_d^0/\mathsf{X}_d^0},$$

the Hodge bundle of the universal curve over X_d^0.

Since the map $u : x \mapsto ux$ induces an isomorphism

$$u : C_{u\cdot F} \xrightarrow{\sim} C_F$$

it has a natural action $u^* : \Omega^1[C_F] \to \Omega^1[C_{u\cdot F}]$ on the differentials and hence, on the sections of $\underline{\alpha}^h$, for $h \in \mathbb{Z}$. More specifically, if $s \in \Gamma(\mathsf{X}_d^0, \underline{\alpha}^{\otimes h})$, one can write $s = \Phi \cdot \eta^{\otimes h}$ with $\Phi \in \mathcal{O}(\mathsf{X}_d^0)$; for $F \in \mathsf{X}_d^0$, one has

$$u^*s(F) = \Phi(F) \cdot (u^*\eta(F))^{\otimes h}.$$

LEMMA 3.2.1. *The section $\eta \in \Gamma(\mathsf{X}_d^0, \underline{\alpha})$ satisfies for $u \in G$ and $F \in \mathsf{X}_d^0$, then*

$$u^*\eta(F) = \det(u)^{w_0} \cdot \eta(u \cdot F), \quad \textit{with } w_0 = \binom{d}{3} = \frac{dg}{3} \in \mathbb{N}.$$

Proof. Since $\dim \underline{\alpha}[F] = \dim \underline{\alpha}[u \cdot F] = 1$, there is $c(u, F) \in k^\times$ such that

$$u^*\eta(F) = c(u, F) \cdot \eta(u \cdot F).$$

and c is a "crossed character", satisfying

$$c(uu', F) = c(u, F)\, c(u', u \cdot F).$$

Now the regular function $F \mapsto c(u, F)$ does not vanishes on X_d^0. By Lemma 3.2.2 below and the irreducibility of the discriminant (Prop. 3.1.1), we have

$$c(u, F) = \chi(u)(Disc F)^{n(u)}$$

with $\chi(u) \in k^\times$ and $n(u) \in \mathbb{Z}$. The group G being connected, the function $n(u) = n$ is constant. Since $c(\mathbf{I}_3, F) = 1$, we have $(Disc F)^n = \chi(\mathbf{I}_3)^{-1}$, and this implies $n = 0$. Hence, $c(u, F)$ is independent of F and χ is a character of G. Since the group of commutators of G is $SL_3(k)$, we have

$$\chi(u) = \det(u)^{w_0}$$

for some $w_0 \in \mathbb{Z}$. It is therefore enough to compute $\chi(u)$ when $u = \lambda\mathbf{I}_3$, with $\lambda \in k^\times$. In this case $u \cdot F = \lambda^d F$. Moreover, for all $f \in X_{d-3}$, since the section η_f is homogeneous of degree -1

$$\eta_f(\lambda^d F) = \lambda^{-d} \cdot \eta_f(F), \text{ and } \eta(\lambda^d F) = \lambda^{-dg} \cdot \eta(F).$$

Hence, as u is the identity on the curve $C_F = C_{u\cdot F}$,

$$u^*\eta(F) = \eta(F) = \lambda^{dg} \cdot \eta(u \cdot F) = \det(u)^{w_0} \cdot \eta(u \cdot F).$$

This implies

$$\det(u)^{w_0} = \lambda^{3w_0} = \lambda^{dg},$$

and the result is proven. □

We made use of the following elementary lemma:

LEMMA 3.2.2. *Let $f \in k[T_1, \ldots, T_n]$ be irreducible and let $g \in k(T_1, \ldots, T_n)$ be a rational function which has neither zeroes nor poles outside the set of zeroes of f. Then there is an $m \in \mathbb{Z}$ and $c \in k^\times$ such that $g = cf^m$.*

Proof. This is an immediate consequence of Hilbert's Nullstellensatz, together with the fact that the ring $k[T_1, \ldots, T_n]$ is factorial. □

For any $h \in \mathbb{Z}$, we denote by $\Gamma(\mathsf{X}_d^0, \underline{\alpha}^{\otimes h})^G$ the subspace of sections $s \in \Gamma(\mathsf{X}_d^0, \underline{\alpha}^{\otimes h})$ such that

$$u^* s(F) = s(u \cdot F) \quad \text{for every } u \in G, F \in \mathsf{X}_d^0.$$

PROPOSITION 3.2.3. *Let $h \geqslant 0$ be an integer. The linear map*

$$\Phi \mapsto \rho(\Phi) = \Phi \cdot \eta^{\otimes h}$$

is an isomorphism

$$\rho : \mathsf{Inv}_{gh}(\mathsf{X}_d^0) \xrightarrow{\sim} \Gamma(\mathsf{X}_d^0, \underline{\alpha}^{\otimes h})^G.$$

Proof. Let $\Phi \in \mathsf{Inv}_{gh}(\mathsf{X}_d^0)$, $s = \rho(\Phi) = \Phi \cdot \eta^{\otimes h}$, and $w = dgh/3$, the weight of Φ. Then using Lem.3.2.1,

$$\begin{aligned} u^* s(F) &= \Phi(F) \cdot (u^* \eta(F))^{\otimes h} \\ &= \Phi(F) \cdot \det(u)^{w_0 h} \cdot \eta(u \cdot F)^{\otimes h} \\ &= \det(u)^w \Phi(F) \cdot \eta(u \cdot F)^{\otimes h} \\ &= \Phi(u \cdot F) \cdot \eta(u \cdot F)^{\otimes h} = s(u \cdot F). \end{aligned}$$

Hence, $\rho(\Phi) \in \Gamma(\mathsf{X}_d^0, \underline{\lambda}^{\otimes h})^G$. Conversely, the inverse of ρ is the map $s \mapsto s/\eta^{\otimes h}$, and this proves the proposition. □

3.3 Modular forms as invariants

Let $d > 2$ be an integer and $g = \binom{d}{2}$. Since the fibres of $\mathsf{Y}_d^0 \longrightarrow \mathsf{X}_d^0$ are nonsingular non hyperelliptic plane curves of genus g, by the universal property of M_g we get a morphism

$$p : \mathsf{X}_g^0 \longrightarrow \mathsf{M}_g^0,$$

where M_g^0 is the moduli stack of nonhyperelliptic curves of genus g and $p^* \underline{\lambda} = \underline{\alpha}$ by construction. This induces a morphism

$$p^* : \Gamma(\mathsf{M}_g^0, \underline{\lambda}^{\otimes h}) \longrightarrow \Gamma(\mathsf{X}_d^0, \underline{\alpha}^{\otimes h}).$$

Moreover, for $u \in G$, since $u : C_{u \cdot F} \to C_F$ is an isomorphism, we get the following commutative diagram

$$\begin{array}{ccc} \underline{\lambda}[C_F] & \xrightarrow{u^*} & \underline{\lambda}[C_{u \cdot F}] \\ {\scriptstyle p^*}\big\downarrow & & {\scriptstyle p^*}\big\downarrow \\ \underline{\alpha}[F] & \xrightarrow{u^*} & \underline{\alpha}[u \cdot F]. \end{array}$$

For any $f \in \Gamma(\mathsf{M}_g^0, \underline{\lambda}^{\otimes h})$, the modular invariance of f means that

$$u^* f(C_F) = f(C_{u \cdot F}).$$

Then

$$u^*[(p^* f)(F)] = u^*[p^*(f(C_F))] = p^*[u^* f(C_F)] = p^*[f(C_{u \cdot F})] = (p^* f)(u \cdot F),$$

and this means that $p^*f \in \Gamma(\mathsf{X}_d^0, \underline{\alpha}^{\otimes h})^G$. Combining this result with Prop.3.2.3, we obtain:

Proposition 3.3.1. *For any integer $h \geqslant 0$, the linear map $\sigma = \rho^{-1} \circ p^*$ is a homomorphism:*

$$\Gamma(\mathsf{M}_g^0, \underline{\lambda}^{\otimes h}) \longrightarrow \mathsf{Inv}_{gh}(\mathsf{X}_d^0)$$

such that

$$\sigma(f)(F) = f(C_F, \lambda)$$

with $\lambda = (p^*)^{-1}\eta$, for any $F \in \mathsf{X}_d^0$ and any section $f \in \Gamma(\mathsf{M}_g^0, \underline{\lambda}^{\otimes h})$. □

We finally make a link between invariants and analytic Siegel modular forms. Let $F \in X_d^0(\mathbb{C})$ and let $\eta_1, \dots, \eta_g$ be the basis of regular differentials on C_F defined in Sec.3.2. Let $\gamma_1, \dots \gamma_{2g}$ be a symplectic basis of $H_1(C, \mathbb{Z})$ (for the intersection pairing). The matrix

$$\Omega = [\Omega_1 \ \Omega_2] = \begin{pmatrix} \int_{\gamma_1} \eta_1 & \cdots & \int_{\gamma_{2g}} \eta_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \eta_g & \cdots & \int_{\gamma_{2g}} \eta_g \end{pmatrix}$$

belongs to the set $\mathcal{R}_g \subset \mathbf{M}_{g,2g}(\mathbb{C})$ of Riemann matrices, and $\tau = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_g$.

Corollary 3.3.2. *Let $f \in \mathbf{S}_{g,h}(\mathbb{C})$ be a geometric Siegel modular form, $\widetilde{f} \in \mathbf{R}_{g,h}(\mathbb{C})$ the corresponding analytic modular form, and $\Phi = \sigma(\theta^* f)$ the corresponding invariant. In the above notation,*

$$\Phi(F) = (2i\pi)^{gh} \frac{\widetilde{f}(\tau)}{\det \Omega_2^h} .$$

Proof. Let $\lambda = (p^*)^{-1}(\eta)$ and $\omega = (\theta^*)^{-1}(\lambda)$. From Prop.2.3.1 and 3.3.1, we deduce

$$\Phi(F) = (\theta^* f)(C_F, \lambda) = f(JacC_F, \omega).$$

On the other hand, by the canonical identifications

$$\Omega^1[C_F] = \Omega^1[JacC_F], \quad H_1(C_F, \mathbb{Z}) = H_1(JacC_F, \mathbb{Z})$$

and Prop.2.4.4 we get

$$f(JacC_F, \omega) = (2i\pi)^{gh} \frac{\widetilde{f}(\tau)}{\det \Omega_2^h} ,$$

from which the result follows. □

4. The case of genus 3

4.1 Klein's formula

We recall the definition of theta functions with (entire) characteristics

$$[\underline{\varepsilon}] = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \in \mathbb{Z}^g \oplus \mathbb{Z}^g,$$

following [2]. The *(classical) theta function* is given, for $\tau \in \mathbb{H}_g$ and $z \in \mathbb{C}^g$, by

$$\theta \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} (z, \tau) = \sum_{n \in \mathbb{Z}^g} q^{(n+\varepsilon_1/2)\tau(n+\varepsilon_1/2)+2(n+\varepsilon_1/2)(z+\varepsilon_2/2)}.$$

The *Thetanullwerte* are the values at $z = 0$ of these functions, and we write

$$\theta[\underline{\varepsilon}](\tau) = \theta \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} (\tau) = \theta \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} (0, \tau).$$

Recall that a characteristic is *even* if $\varepsilon_1 \cdot \varepsilon_2 \equiv 0 \pmod 2$ and *odd* otherwise. Let S_g (resp. U_g) be the set of even characteristics with coefficients in $\{0,1\}$. For $g \geqslant 2$, we put $h = \#S_g/2 = 2^{g-2}(2^g+1)$ and

$$\widetilde{\chi}_h(\tau) = (2i\pi)^{gh} \prod_{\varepsilon \in S_g} \theta[\varepsilon](\tau).$$

In his beautiful paper [17], Igusa proves the following result [*loc. cit.*, Lem. 10 and 11]. Denote by $\widetilde{\Sigma}_{140}$ the modular form defined by the thirty-fifth elementary symmetric function of the eighth power of the even Thetanullwerte. Recall that a principally polarized abelian variety (A, a) is decomposable if it is a product of principally polarized abelian varieties of lower dimension, and indecomposable otherwise.

Theorem 4.1.1. *If $g \geqslant 3$, then $\widetilde{\chi}_h(\tau) \in \mathbf{R}_{g,h}(\mathbb{C})$. Moreover, If $g = 3$ and $\tau \in \mathbb{H}_3$, then:*

(i) *A_τ is decomposable if $\widetilde{\chi}_{18}(\tau) = \widetilde{\Sigma}_{140}(\tau) = 0$.*
(ii) *A_τ is a hyperelliptic Jacobian if $\widetilde{\chi}_{18}(\tau) = 0$ and $\widetilde{\Sigma}_{140}(\tau) \neq 0$.*
(iii) *A_τ is a non hyperelliptic Jacobian if $\widetilde{\chi}_{18}(\tau) \neq 0$.* □

Using Prop. 2.2.1, we define the geometric Siegel modular form of weight h

$$\chi_h(A_\tau) = (2i\pi)^{gh}\, \widetilde{\chi}_h(\tau)(dz_1 \wedge \cdots \wedge dz_g)^{\otimes h}.$$

Ichikawa [15], [16] proved that $\chi_h \in \mathbf{S}_{g,h}(\mathbb{Q})$. For $g = 3$, one finds

$$\chi_{18}(A_\tau) = -(2\pi)^{54}\, \widetilde{\chi}_{18}(\tau)(dz_1 \wedge dz_2 \wedge dz_3)^{\otimes 18}.$$

Now we are ready to give a proof of the following result [20, Eq. 118, p. 462]:

Theorem 4.1.2 Klein's formula. *Let $F \in \mathsf{X}_4^0(\mathbb{C})$ and C_F be the corresponding smooth plane quartic. Let η_1, η_2, η_3 be the classical basis of $\Omega^1[C_F]$ from Sec.3.2 and $\gamma_1, \ldots \gamma_6$ be a symplectic basis of $H_1(C_F, \mathbb{Z})$ for the intersection pairing. Let*

$$\Omega = [\Omega_1\ \Omega_2] = \begin{pmatrix} \int_{\gamma_1} \eta_1 & \cdots & \int_{\gamma_6} \eta_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \eta_3 & \cdots & \int_{\gamma_6} \eta_3 \end{pmatrix}$$

be a period matrix of $Jac(C)$ and $\tau = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_3$. Then

$$Disc(F)^2 = \frac{1}{2^{28}}(2\pi)^{54}\frac{\widetilde{\chi}_{18}(\tau)}{\det(\Omega_2)^{18}}.$$

Proof. Cor.3.3.2 shows that $I = \sigma \circ \theta^*(\chi_{18})$satisfies for any $F \in \mathsf{X}_4^0$,

$$I(F) = -(2\pi)^{54}\frac{\widetilde{\chi}_{18}(\tau)}{\det \Omega_2^{18}}.$$

Moreover Th. 4.1.1 (iii) shows that $I(F) \neq 0$ for all $F \in \mathsf{X}_4^0$. Thus I is a non-zero invariant of weight 54. Applying Lem. 3.2.2 for the discriminant, we find by comparison of the weights that $I = cDisc^2$ with $c \in \mathbb{C}$ a constant. But if F_m is the Ciani quartic associated to a matrix $m \in \mathrm{Sym}_3(k)$ as in Example 3.1.2, it is proven in [21, Cor. 4.2] that Klein's formula is true for F_m and $c = -2^{28}$. □

Remark 4.1.3. The morphism θ^* defines an injective morphism of graded k-algebras

$$\mathbf{S}_3(k) = \oplus_{h\geqslant 0}\mathbf{S}_{3,h}(k) \longrightarrow \mathbf{T}_3(k) = \oplus_{h \geqslant 0}\mathbf{T}_{3,h}(k).$$

In [14], Ichikawa proves that if k is a field of characteristic 0, then $\mathbf{T}_3(k)$ is generated by the image of $\mathbf{S}_3(k)$ and a primitive Teichmüller form $\mu_{3,9} \in \mathbf{T}_{3,9}(\mathbb{Z})$ of weight 9, which is not of Siegel modular type. He also proves in [16] that

$$\theta^*(\chi_{18}) = -2^{28} \cdot (\mu_{3,9})^2. \tag{2}$$

Th. 4.1.2 implies that $\mu_{3,9}$ is actually equal to the discriminant up to a sign. This might probably be deduced from the definition of $\mu_{3,9}$, although we did not sort it out (see also [18, Sec. 2.4]).

Remark 4.1.4. Besides [23] and [11] where an analogue of Klein's formula is derived in the hyperelliptic case, there exists a beautiful algebraic Klein's formula, linking the discriminant with irrational invariants [9, Th.11.1].

4.2 Jacobians among abelian threefolds

Let $k \subset \mathbb{C}$ be a field and let $g = 3$. We prove the following theorem which allows us to determine whether a given abelian threefold defined over k is k-isomorphic to a Jacobian of a curve defined over the same field. This settles the question of Serre recalled in the introduction.

Theorem 4.2.1. *Let (A, a) be a principally polarized abelian threefold defined over $k \subset \mathbb{C}$. Let $\omega_1, \omega_2, \omega_3$ be a basis of $\Omega^1_k[A]$ and $\gamma_1, \ldots \gamma_6$ a symplectic basis of $H^1(A, \mathbb{Z})$, in such a way that*

$$\Omega = [\Omega_1 \ \Omega_2] = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_6} \omega_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \omega_3 & \cdots & \int_{\gamma_6} \omega_3 \end{pmatrix}$$

is a period matrix of (A, a). Put $\tau = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_3$.

(i) *If $\widetilde{\Sigma}_{140}(\tau) = 0$ then (A, λ) is decomposable. In particular it is not a Jacobian.*

(ii) *If $\widetilde{\Sigma}_{140}(\tau) \neq 0$ and $\widetilde{\chi}_{18}(\tau) = 0$ then there exists a hyperelliptic curve X/k such that $(JacX, j) \simeq (A, a)$.*

(iii) *If $\widetilde{\chi}_{18}(\tau) \neq 0$ then (A, a) is isomorphic to a Jacobian if and only if*

$$-\chi_{18}(A, \omega_1 \wedge \omega_2 \wedge \omega_3) = (2\pi)^{54} \frac{\widetilde{\chi}_{18}(\tau)}{\det(\Omega_2)^{18}}$$

is a square in k.

Proof. The first and second points follow from Th.4.1.1 and Th.1.1.1. Suppose now that (A, a) is isomorphic over k to the Jacobian of a non hyperelliptic genus 3 curve C/k and let $\omega = \omega_1 \wedge \omega_2 \wedge \omega_3$. Using Prop.2.3.1, we get

$$-\chi_{18}(A, \omega) = \theta^*(-\chi_{18})(C, \lambda)$$

with $\lambda = \theta^*\omega$. The left hand side is (Prop.2.4.4)

$$-\chi_{18}(A, \omega) = -(2i\pi)^{54} \frac{\widetilde{\chi}_{18}}{\det(\Omega_2)^{18}} = (2\pi)^{54} \frac{\widetilde{\chi}_{18}(\tau)}{\det(\Omega_2)^{18}}.$$

According to Rem.4.1.3, the right hand side of the equality is

$$\theta^*(-\chi_{18})(C, \lambda) = 2^{28} \cdot \mu_{3,9}^2(C, \lambda) = (2^{14} \cdot \mu_{3,9}(C, \lambda))^2$$

so the desired expression is a square in k. On the contrary, Cor.2.4.3 shows that if (A, a) is a quadratic twist of a Jacobian (A', a') then there exists a non square $c \in k$ such that

$$-\bar{\chi}_{18}(A) = c^9 \cdot (-\bar{\chi}_{18}(A')).$$

As we have just proved that $-\bar{\chi}_{18}(A')$ is a non-zero square in $k/k^{\times 18}$, $-\bar{\chi}_{18}(A)$ (and then $-\chi_{18}(A, \omega)$) is not. $\square$

Corollary 4.2.2. *In the notation of Th.4.2.1, the quadratic character ε of $Gal(k_{\rm sep}/k)$ introduced in Theorem 1.1.1 is given by $\varepsilon(\sigma) = d/d^\sigma$, where*

$$d = \sqrt{(2\pi)^{54}\frac{\widetilde{\chi}_{18}(\tau)}{\det(\Omega_2)^{18}}}\,,$$

with an arbitrary choice of the square root.

4.3 Beyond genus 3

It is natural to try to extend our results to the case $g > 3$. The first question to ask is

— Does there exist an analogue of Klein's formula for $g > 3$?

Here we can give a partial answer. Using Sec.2.3, we can consider the Teichmüller modular form $\theta^*(\chi_h)$ with $h = 2^{g-2}(2^g+1)$. In [16, Prop.4.5] (see also [29]), it is proven that for $g > 3$ the element

$$\theta^*(\chi_h)/2^{2^{g-1}(2^g-1)}$$

has as a square root a primitive element $\mu_{g,h/2} \in \mathbf{T}_{g,h/2}(\mathbb{Z})$. If $g = 4$, in the footnote, p. 462 in [20] we find the following amazing formula

$$\frac{\widetilde{\chi}_{68}(\tau)}{\det(\Omega_2)^{68}} = c \cdot \Delta(X)^2 \cdot T(X)^8. \qquad (3)$$

Here $\tau = \Omega_2^{-1}\Omega_1$, with $\Omega = [\Omega_1\ \Omega_2]$ a period matrix of a genus 4 non hyperelliptic curve X given in $\mathbb{P}^3$ as an intersection of a quadric Q and a cubic surface E. The elements $\Delta(X)$ and $T(X)$ are defined in the classical invariant theory as, respectively, the discriminant of Q and the tact invariant of Q and E (see [26, p.122]). No such formula seems to be known in the non hyperelliptic case for $g > 4$.

Let us now look at what happens when we try to apply Serre's approach for $g > 3$. To begin with, when g is even, we cannot use Cor.2.4.2 to distinguish between quadratic twists. In particular, using the previous result, we see that $\chi_h(A, \omega_k)$ is a square when A is a principally polarized abelian variety defined over k which is geometrically a Jacobian. A natural question is:

— What is the relation between this condition and the locus of geometric Jacobians over k?

Let us assume now that g is odd. Corollary 2.4.3 shows that there exists $c \in k \setminus k^2$ such that

$$\overline{\chi}_h(A') = c^{h/2} \cdot \overline{\chi}_h(A)$$

for a Jacobian A and a quadratic twist A'. What enabled us to distinguish between A and A' when $g = 3$ is that $h/2 = 9$ is odd. However as soon as $g > 3$, $2 \mid 2^{g-3}$, the power $g-3$ being the maximal power of 2 dividing $h/2$, so it is not enough for $\bar{\chi}_{18}(A)$ to be a square in k to make a distinction between A and A'. It must rather be an element of $k^{2^{g-2}}$. It can be easily seen from the proof of [29, Th.1] that $\theta^*(\chi_h)$ does not admit a fourth root. According to [1] or [30] this implies $\overline{\chi}_h(A) \notin k^{2^{g-2}}$ for infinitely many Jacobians A defined over number fields k. So we can no longer use the modular form χ_h to easily characterize Jacobians over k. The question is:

— Is it possible to find a modular form to replace χ_h in our strategy when $g > 3$?

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