The report of Mikhail Bondarko for 2007

1 Scientific activity; publications

The main scientific goal of the project is the study of categories of motives. Recall that (Voevodsky's) motives correspond to a certain "universal cohomology theory for smooth algebraic varieties". Note that in contrast with "classical"categories of motives (i.e. Chow, homological and numerical ones) Voevodsky's $DM_{gm}^{eff} \subset DM_{gm}$ are endowed with natural (and highly non-trivial) structures of triangulated categories. One of my main results is the study of the relation of DM_{gm} with $Chow$; this sheds some light on the celebrated *standard* motivic conjectures.

In 2007 I finished my preprint **Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky vs. Hanamura** (see http://arxiv.org/abs/math.AG/0601713); it was accepted for publication in the Journal of the Institute of Mathematics of Jussieu. It contains a full description of Voevodsky's DM_{gm}^{eff} in terms of 'twisted' Suslin cubical complexes (in the sense of Kapranov and Bondal). In particular, for any motivic complex M (for instance, the Suslin complex of an arbitrary variety) there exists a quasi-isomorphic complex M' 'constructed from' the Suslin complexes of smooth projective varieties; M' is unique up to a homotopy. I also proved the following results.

Theorem 1.1. *I There exist a conservative exact weight complex functor* $t : DM_{gm}^{eff} \subset DM_{gm} \to K^b(Chow^{eff}) \subset K^b(Chow)$.

II t *induces isomorphisms* $K_0(DM_{gm}^{eff}) \to K_0(Chow)$ *and* $K_0(DM_{gm}) \to K_0(Chow)$*; they are isomorphisms of rings.*

III For any cohomological realization $H : DM_{gm} \to D_B(A)$ *(here* A *is an abelian category and* $X \in Obj DM_{gm}^{eff}$*) there exists a natural weight spectral sequence* $S : H^i(P_{-j}) \to H^{i+j}(X)$ *where* (P_i) *is a representative of* $t(X)$. S *is canonical and motivically functorial starting from* E_1*. It yields the usual weight spectral sequences and weight filtrations for mixed Hodge and étale cohomology of varieties.*

IV Voeovodsky's $DM_{gm} \otimes \mathbb{Q}$ *is antiequivalent to the Hanamura's motivic category.*

V A motif (an object of Voevodsky's DM_{gm}*) is a mixed Tate one whenever its weight complex is.*

A new method of attaching weights to cohomology functors was developed. In particular, a certain weight filtration for motivic cohomology was defined; note that this filtration is non-trivial, new, and universal for the important class of Bloch-Ogus cohomology theories.

I also wrote a preprint **Weight structures, weight filtrations, weight spectral sequences, and weight complexes for triangulated categories (including motives and spectra)** (electronic, http://arxiv.org/abs/0704.4003).

I showed that parts I-III of Theorem 1.1 follow from a very general relevant formalism for triangulated categories; this situation was not described in literature. One considers a set of axioms that are (in a certain sense) "dual" to the axioms of t-structures; I call this a weight structure. Several properties of weight structures are similar to those of t-structures; yet other ones are quite distinct.

Each triangulated category C with a weight structure has an additive heart with the property that there are no morphisms of positive degrees between objects of the heart in C. Any weight structure defines a conservative 'weight complex' functor to a certain "weak" homotopy category of complexes over the heart. Moreover, the weight structure gives a Postnikov tower of any object which is canonical and functorial up to homotopy. In particular, for any (co)homological functor one obtains a 'weight spectral sequence' whose terms are (co)homology of the corresponding objects of the heart; this spectral sequence is canonical and functorial starting from E_2. Next, one can often obtain t-structures and weight structures from each other by passing to (left and right) 'adjacent subcategories'. The hearts of 'adjacent' structures are closely connected with each other.

The most important examples of this formalism are Voevodsky's $DM_{gm} \subset DM_{gm}^{eff}$ and the stable homotopy category SH (of spectra).

One of the main consequences of the weight structure formalism for motives is that canonical weight filtrations and spectral sequences exist for arbitrary realizations of motives (not necessarily having a differential graded enhancement). In particular, weights should exist for the (conjectural!) "mixed motivic"cohomology (of varieties and motives).

The adjacent structure formalism yields that Voevodsky's DM_{-}^{eff} has a t-structure whose heart is the category of additive functors $Chow^{eff} \to Ab$.

I also proved that (a certain version of) the weight complex functor can be defined on $DM_{gm}^{eff} \subset DM_{-}^{eff}$ without using the resolution of singularities (so one can define it for motives over any perfect field).

For spectra the weight spectral sequence specializes to the Atiyah-Hirzebruch sequence. The formalism allows to calculate $K_0(SH_{fin})$ and certain $K_0(\operatorname{End} SH_{fin})$ (and $K_0(\operatorname{End}^n SH_{fin})$ for $n \in \mathbb{N}$); here SH_{fin} is the category of finite spectra.

The results on adjacent structure establishes (and allows to study) a connection of the coniveau filtration on cohomology of motives with the (Voevodsky's) homotopy t-structure on DM_{-}^{eff} (this extends the seminal result of Bloch and Ogus). In particular, torsion motivic cohomology of motives can be expressed in terms certain étale cohomology (here the recently proved Beilinson-Lichtenbaum conjecture is used). In section 2.2 of the new preprint **Artin's Vanishing for torsion motivic homology; numerical motives form a tannakian category** (see http://arxiv.org/abs/0711.3918) a nice formula of this sort (and also a formula for the "difference"of the motivic cohomology with the étale one) is proved.

In the latter preprint several interesting motivic problems are studied. Unfortunately, Theorem 2.1.1 (of version 2 of the preprint) is wrong in the form it is stated. This makes (most of) the results of the preprint conditional modulo Theorem 2.3.2. Yet I hope to correct the proof of Theorem 2.3.2. Note also that the rational version of it follows from certain "standard"motivic conjectures; hence the preprint (at least) reveals certain new connections between motivic conjectures.

This preprint is also related with a short preprint **Explicit generators for (conjectural) mixed motives (in Voevodsky's DM_{gm}^{eff}). The Kunneth decomposition of pure (numerical) motives**, http://arxiv.org/abs/math/0703499. In the latter preprint I showed (briefly) that if certain "standard"conjectures are fulfilled then the Kunneth decomposition of the diagonal and a certain generating set of mixed motives could be described quite explicitly.

In 2007 I made two talks at international mathematical conferences on the topics described.

1. Arithmetic Geometry, June 13–19, 2007, Saint-Petersburg, Russia: "The universal Euler characteristic for motives".

2. International Algebraic Conference dedicated to the 100th anniversary of D. K. Faddeev, September 24–29, 2007, Saint-Petersburg, Russia: "Weights for cohomology: weight structures, filtrations, spectral sequences, and weight complexes (for motives and spectra)".

2 Pedagogical activity

In 2007 I led student's practice in higher algebra and number theory and read lectures on this subject (in St. Petersburg State University). Besides I actively participated in the composition of a book of problems in Number theory; it will be published and used for teaching students.