The report of Mikhail Bondarko for 2009

1 Scientific activity

The main scientific goal of my project is the study of various categories of motives. Recall that (Voevodsky's) motives correspond to a certain 'universal cohomology theory for smooth algebraic varieties'. Note that in contrast with 'classical' categories of motives (i.e. Chow, homological and numerical ones) Voevodsky's $DM_{gm}^{eff} \subset DM_{gm}$ are endowed with natural (and highly non-trivial) structures of triangulated categories. One of the main results published in 2009 is the construction of a close relation of DM_{gm} with $K^b(Chow)$ via a certain exact conservative weight complex functor. I also proved (in 2005-2009) several other results on motives: they concern coniveau spectral sequences, motivic cohomology, and unramified cohomology. One of the main methods was the application of weight structures in triangulaed categories (that I defined in 2007); in 2009 I developed this technique further.

In 2009 my paper [1] was published. The contents of the paper were described in my previous reports.

My preprint [2] was updated and accepted by the Journal of K-theory. Recall that in this paper a new general formalism for triangulated categories was described; see [4] for a survey of the subject. I considered a set of axioms that are (in a certain sense) 'almost dual' to the axioms of t-structures; I called this a weight structure. The easiest example of a weight structure could be described in terms on stupid truncations for the homotopy category of complexes over any additive category. Several properties of weight structures are similar to those of t-structures; yet other ones are quite distinct. A weight structure gives a Postnikov tower of any object X which is canonical and functorial 'up to cohomology zero maps'. It follows that for any (co)homological functor H one obtains a *weight spectral sequence* (weakly) conveging to $H(X)$, whose terms are (co)homology of the corresponding objects of the *heart of the weight structure* (in the case of the Chow weight structure for $DM_{gm}^{eff} \subset DM_{gm}$ the heart is the category of Chow motives). This spectral sequence is canonical and functorial starting from E_2.

I also proved that weight structures could be 'glued' in a manner that is similar to those for t-structures (see §8.2 of [2]). This fact could be used to 'almost construct' a series of weight structures for the category of comotives (that was introduced in [3]; see §4.8 of [3] and §9.3 of [4]). This series is indexed by a single integral parameter; all the structures induce the same weight structure on the category of birational comotives, and for the i-th weight structure tensoring by $\mathbb{Z}(1)[i]$ is *weight-exact* (i.e. it sends 'positive' comotives to positive ones and 'negative' comotives to negative ones). In particular, for $i = 2$ one obtains the Chow weight structure; for $i = 1$ one obtains the Gersten weight structures. These structures correspond to 'the usual weight' and coniveau spectral sequences for cohomology of (co)motives, respectively. It is quite amazing that spectral sequences that are so distinct from the geometric point of view differ just by [1] (in this description)! Possibly, other members of this series could be also interesting (especially the one corresponding to $i = 0$).

In 2009 several other topics were also added to [3] (already mentioned in the previous report; in 2009 the volume of the preprint doubled). In the case of countable k, it describes a 'triangulated analogue' of the coniveau spectral sequence. To this end I constructed a so-called 'Gersten' weight structure. Since Voevodsky's categories $DM_-^{eff} \supset DM_{gm}^{eff}$ do not contain any motives for infinite field extensions of k (which should generate the heart of this weight structure), I construct a certain new triangulated category of *comotives* $\mathfrak{D}$ that contains Voevodsky's DM_{gm}^{eff} as well as (co)motives for arbitrary projective limits of smooth schemes (this is a completion of DM_{gm}^{eff} with respect to certain homotopy limits). In this category the motif of a variety was 'decomposed' (in the sense of Postnikov towers) into Tate twists of comotives of its points. The theory of weight structures yields that this result could be generalized to arbitrary objects of DM_{gm}^{eff}. So, one obtains a weight spectral sequence with respect to this weight structure for any comotif X and any cohomology theory $\mathfrak{D} \to \underline{A}$. In the case when H is a theory extended from DM_{gm}^{eff} (via a certian method described in ibid.) and X is the (co)motif of a smooth variety, one obtains the 'usual' coniveau spectral sequence. One obtains that coniveau spectral sequences (starting from E_2) are functorial with respect to morphisms in $\mathfrak{D} \supset DM_{gm}^{eff}$; this is quite difficult to see from the 'basic' definition of coniveau spectral sequences.

Among the new topics is the theory of *nice dualities* of triangulated categories (those are pairings of two distinct triangulated categories with values in an abelian category); this setting is new and could be interesting for itself. Let $\Phi : \underline{C}^{op} \times \underline{D} \to \underline{A}$ be a nice duality; let $\underline{C}$ be endowed with a weight structure w, $\underline{D}$ be endowed with a t-structure t, and w be *orthogonal* to t (with respect to Φ); for some $Y \in Obj\underline{D}$ we consider the functor $H = \Phi(-, Y)$. Then one has a functorial description of the weight spectral sequence for H (starting from E_2) in terms of t-truncations of Y; see Theorem 2.6.1 of [3]. This is a powerful tool for comparing spectral sequences (in this situation); it does not require constructing any complexes (and filtrations for them) in contrast to the method of Paranjape (probably, originating from Deligne). In particular, the technique of orthogonal structures establishes (and allows to study) a connection of the coniveau filtration on cohomology of (co)motives with the (Voevodsky's) homotopy t-structure on DM_-^{eff}. This generalizes to motives (and extends) the seminal coniveau spectral sequence computations of Bloch and Ogus.

Other new topics are: comparison of weight spectral sequences for distinct weight structure (this allows to compare coniveau spectral sequences over distinct fields, and also compare coniveu spectral sequences with weight spectral sequences corresponding to the Chow weight structure); the Gysin distinguished triangle for comotives of schemes not (necessarily) of finite type over k.

I also proved some results on motives with finite coefficients and Artin-Tate motives.

An adjont to the 'projection' functor from the category of integral Voevodsky's motives to the category of motives with finite coefficients was constructed.

The category of triangulated Tate motives with finite coefficients over an algebraically closed field was calculated completely. I also proved that the category of Artin-Tate motives could be endowed with a weight structure compatible with the Gersten weight structure for comotives; it follows that coniveau spectral sequences for cohomology of such motives have 'simple and economic description'.

The publications (supported by the Balzan foundation) are listed in the end of the report.

2 Pedagogical activity

In 2009 I led student's practice in higher algebra and number theory (in St. Petersburg State University). Besides I actively participated in the composition of two books of problems: a one in Number theory and a one in Field theory. The first one is now it is published.

3 Conferences

In 2009 I made talks at the following conferences:

1. Young Mathematics in Russia, Moscow, 12–13.01.2009.

2. Workshop "Finiteness for Motives and Motivic Cohomology Regensburg, 9–13.02.2009.

3. Workshop on Motivic Homotopy Theory, Münster, 27–31.07.2009.

4. Algebraic Conference dedicated to the 60th Anniversary of A. I. Generalov, St. Petersburg, 2–3.09.2009.

Список литературы

[1] Bondarko M.V., Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky vs. Hanamura// J. of the Inst. of Math. of Jussieu, v.8 (2009), no. 1, 39–97, see also **http://arxiv.org/abs/math.AG/0601713**.

[2] Bondarko M., Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general), to appear in J. of K-theory, **http://arxiv.org/abs/0704.4003**.

[3] Bondarko M., Motivically functorial coniveau spectral sequences; direct summands of cohomology of function fields, preprint, **http://arxiv.org/abs/0812.2672**.

[4] Bondarko M.V., Weight structures and motives; comotives, coniveau and Chow-weight spectral sequences: a survey, preprint, **http://arxiv.org/abs/0903.0091**.