Dynasty Fellowship Report 2012

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1. Main results

In 2012, together with Klaus Altmann and Lars Petersen we obtained a comparison result between colored fans of spherical varieties on one side and divisorial fans on wonderful compactifications on the other side [AKP]. I also studied applications of convex geometric divided difference operators to Schubert calculus and Newton–Okounkov convex bodies of Bott–Samelson resolutions [K1,K2]. Below I describe these results in more detail. Together with Pavel Gusev and Vladlen Timorin we obtained formulas for generating functions of the number of vertices in Gelfand–Zetlin polytopes for 3-step flag varieties [GKT].

1.1. **Colored and divisorial fans.** We relate the language of *colored fans* (combinatorial objects describing spherical varieties) and the language of *polyhedral divisors* (partially combinatorial and partially algebro-geometric objects describing varieties with a torus action). Colored fans are usual fans together with an additional combinatorial data (colors), namely, some rays in the fan can be colored. Polyhedral divisors are finite linear combinations of usual divisors on a variety with coefficients being (possibly unbounded) convex polyhedra. Polyhedral divisors describe affine varieties with a torus action.

More precisely, given an affine variety X with an effective action of a torus $T = (\mathbb{C}^*)^n$ one can construct a polyhedral divisor

$$\mathcal{D}^X = \sum_{D \subset Y} P_D \otimes D$$

on the Chow quotient $Y := X/\!\!/^{\mathrm{Ch}} T$. Here D runs over prime divisors on Y, and coefficients P_D are convex polyhedra in the affine space $N_{\mathbb{R}} := N \otimes \mathbb{R} \simeq \mathbb{R}^n$, where N denotes the lattice of one-parametric subgroups of T. All polyhedra P_D have the same *tail cone* σ (the *tail cone* of a polyhedron $P \subset \mathbb{R}^n$ consists of all vectors $a \in \mathbb{R}^n$ such that $a + P \subset P$, in particular, the tail cone of a bounded polyhedron is empty). Note that polyhedra with the tail cone σ form a semigroup under Minkowski sum, and σ is the neutral element in this semigroup. In particular, adding or suppressing summands of the form $\sigma \otimes D$ does not change $\mathcal{D}^X$. We also allow $\emptyset$ as a coefficient, namely, the summand $\emptyset \otimes D$ means that the polyhedral divisor $\mathcal{D}^X$ should be considered not on Y but on $Y \setminus D$. Let N^* denote the dual lattice of N, i.e. the character lattice of T. For any $u \in \sigma^* \cap N^*$, the polyhedral divisor $\mathcal{D}^X$ yields the usual divisor $\mathcal{D}^X(u)$ on Y as follows:

$$\mathcal{D}^X(u) = \sum_{D \subset Y} \min_{P_D}(u) \cdot D.$$

The affine variety X can be recovered from $\mathcal{D}^X$ as the spectrum of the ring

$$\bigoplus_{u \in \sigma^* \cap N^*} \Gamma(Y \setminus \mathrm{loc}(\mathcal{D}^X), \mathcal{D}^X(u)),$$

where $\mathrm{loc}(\mathcal{D}^X) := \bigcup_{P_D=\emptyset} D$. For instance, if $Y = \mathrm{pt}$ then this is a usual description of affine toric varieties.

To describe arbitrary varieties with a torus action, polyhedral divisors can be glued to form a *divisorial fan*

$$\mathcal{S} = \sum_{D \subset Y} S_D \otimes D$$

in the same way as cones can be glued to form a usual fan. The resulting combinatorial objects S_D are subdivisions of the affine space $N_{\mathbb{R}}$ into polyhedra. The tail cones of polyhedral divisors glue to a fan (called the *tail fan*), and this fan is again the same for all polyhedral subdivisions S_D. An enlightening example of a divisorial fan is given by *toric downgrade*, that is, by regarding a toric variety X of dimension $d > n$ as a variety with the action of a smaller subtorus $T \subset (\mathbb{C}^*)^d$. Then polyhedral subdivisions S_D are slices of the fan of X by n-dimensional subspaces parallel to $N_{\mathbb{R}}$.

Let G be a connected reductive group, and $H \subset G$ a spherical subgroup, that is a Borel subgroup of G acts on G/H (from the left) with an open dense orbit. Recall that a *spherical embedding* of G/H is a normal G-variety with an open dense G-orbit isomorphic to G/H. Spherical embeddings are close relatives of toric varieties. In particular, they have finitely many G-orbits. An important class of spherical varieties consists of *wonderful varieties*, which were recently used by Luna, Cupit-Foutou and others to classify all spherical homogeneous spaces. Wonderful varieties have a unique closed G-orbit (in particular, there are no nontrivial wonderful toric varieties). Arbitrary spherical varieties can be regarded as fibrations over wonderful compactifications with fibers being toric varieties. We use polyhedral divisors to describe exactly where and how this fibration degenerates.

More precisely, we describe spherical embeddings $X \supseteq G/H$ by a divisorial fan $\mathcal{S}$, on a modification Y of the wonderful compactification $\overline{Y} \supseteq G/H'$ with $H' := H \cdot N_G(H)^\circ$. The torus action on X is given by the right action of the torus $T := H/H'$. The set of colors of G/H' (i.e. the set of B-invariant prime divisors) is denoted by $\mathcal{C}(G/H')$, and the valuation given by the color D' is denoted by $\rho(D')$. Using the language of polyhedral divisors allows us to recover the encoded spherical variety directly from the given combinatorial data.

Theorem 1.1 (AKP). *Let $X \supseteq G/H$ be a spherical embedding given by a colored fan Σ^X inside the dual weight lattice $\mathcal{X}^*(G/H)$. If $\mathcal{V}_H \subseteq \mathcal{X}^*(G/H)$ denotes the valuation cone, then X is given by the divisorial fan $\mathcal{S}^X$ on (Y, N) with:*

1) *The base space Y is the toroidal spherical embedding of G/H' given by the (un-) colored fan $(\Sigma_Y, \emptyset)$ arising as the* image fan *of $\Sigma^X \cap \mathcal{V}_H$ via the map $p : \mathcal{X}^*(G/H) \to \mathcal{X}^*(G/H')$. Its rays $a \in \Sigma_Y(1)$ correspond to the G-invariant divisors $D_a \subseteq Y$.*

2) *The maximal cells of the divisorial fan $\mathcal{S} = \mathcal{S}^X$ describing X as a T-variety are labeled by the maximal colored cones $C = (C, \mathcal{F}_C) \in \Sigma^X$ and the elements $w \in W$ of the Weyl group of G. The part of $\mathcal{S}^X$ with label (C, w) is equal to*

$$\mathcal{S}^X(C,w) = \sum_{a \in \Sigma_Y(1)} \mathcal{S}^X_a(C) \otimes D_a + \sum_{D' \in \mathcal{C}(G/H')} (\rho(D') + \mathcal{S}^X_0(C)) \otimes \overline{D'} + \sum_{D' \in \mathcal{C}(G/H') \setminus \mathcal{F}_C} \emptyset \otimes w\overline{D'}$$

where $\mathcal{S}^X_a(C) := C \cap p^{-1}(a)$ is considered as an element of $p^{-1}(a) \cong N_{\mathbb{R}}$.

This theorem reduces to a simpler statement in the case of horospherical varieties (spherical varieties fibered over flag varieties with fibers being toric varieties). In this case, the valuation cone coincides with $N_{\mathbb{R}}$, and G/H' is a flag variety, hence, $Y = \overline{Y} = G/H'$. The coefficients of the divisorial fan $\mathcal{S}^X$ are just shifted copies of the colored fan.

1.2. **Divided difference operators on convex polytopes.** Divided difference (or Demazure) operators play a key role in Schubert calculus and representation theory. Using ideas of [KST], I constructed convex geometric analogs of Demazure operators. Geometric Demazure operators act on polytopes and take a polytope to a polytope of dimension one greater. For instance, Gelfand–Zetlin polytopes can be obtained by applying a suitable composition of geometric Demazure operators to a point.

In 2012, I studied applications of this construction to Schubert calculus for arbitrary reductive groups [K1,K2]. Together with Dave Anderson and Kiumars Kaveh we study relation between (1) string polytopes exhibited by Kiumars as Newton–Okounkov polytopes of complete flag varieties for certain geometric valuations, (2) Newton–Okounkov polytopes for Bott–Samelson resolutions computed by Dave, and (3) polytopes constructed via divided difference operators. For $GL(n)$, each Gelfand–Zetlin polytope can be realized as (1), (2) and (3). For Sp(4), it seems that (1) and (3) always differ, while (2) and (3) are always the same.

Below I give a definition of Demazure operators on convex polytopes and describe polytopes that can be viewed as Newton–Okounkov bodies of flag varieties.

A *root space* of rank n is a coordinate space $\mathbb{R}^d$ together with a direct sum decomposition

$$\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \ldots \oplus \mathbb{R}^{d_n}$$

and a collection of linear functions $l_1, \ldots, l_n \in (\mathbb{R}^d)^*$ such that l_i vanishes on $\mathbb{R}^{d_i}$. We always assume that the summands are coordinate subspaces (so that $\mathbb{R}^{d_1}$ is spanned by the first d_1 basis vectors etc.). The coordinates in $\mathbb{R}^d$ will be denoted by $(x_1^1, \ldots, x_{d_1}^1; \ldots; x_1^n, \ldots, x_{d_n}^n)$.

Let $P \subset \mathbb{R}^d$ be a convex polytope in the root space. It is called a *parapolytope* if for all $i = 1,\ldots, n$, the intersection of P with any parallel translate of $\mathbb{R}^{d_i}$ is a *coordinate parallelepiped*, that is, the parallelepiped

$$\Pi(\mu, \nu) = \{\mu_k \le x_k^i \le \nu_k,\ k = 1, \ldots, d_i\},$$

where $\mu_1, \ldots, \mu_{d_i}, \nu_1, \ldots, \nu_{d_i}$ are real numbers. For instance, if $d = n$ (i.e. $d_1 = \ldots = d_n = 1$) then every polytope is a parapolytope. A less trivial example of a parapolytope is the classical Gelfand–Zetlin polytope Q_λ (where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is an increasing collection of integers) in the root space

$$\mathbb{R}^d = \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-2} \oplus \ldots \oplus \mathbb{R}^1 \qquad (1)$$

of rank $(n-1)$. The polytope Q_λ is given by inequalities $x_j^{i-1} \le x_j^i \le x_{j+1}^{i-1}$ for all $i = 1,\ldots, n-1$ and $1 \le j \le (n-i)$ (we put $x_j^0 = \lambda_j$ for $j = 1,\ldots, n$).

For each $i = 1,\ldots, n$, we now define a *divided difference operator* D_i on parapolytopes. In general, the operator D_i takes values in *convex chains* (or *virtual polytopes*) in $\mathbb{R}^d$.

First, consider the case where $P \subset (c + \mathbb{R}^{d_i})$ for some $c \in \mathbb{R}^d$, i.e. $P = P(\mu, \nu)$ is a coordinate parallelepiped. Choose the smallest $j = 1,\ldots, d_i$ such that $\mu_j = \nu_j$. Define $D_i(P)$ as the coordinate parallelepiped $\Pi(\mu, \nu')$, where $\nu'_k = \nu_k$ for all $k \neq j$ and ν'_j is defined by the equality

$$\sum_{k=1}^{d_i} (\mu_k + \nu'_k) = l_i(c).$$

For an arbitrary parapolytope $P \subset \mathbb{R}^d$ define $D_i(P)$ as the union of $D_i(P \cap (c + \mathbb{R}^{d_i}))$ over all $c \in \mathbb{R}^d$:

$$D_i(P) = \bigcup_{c \in \mathbb{R}^d} \{D_i(P \cap (c + \mathbb{R}^{d_i}))\}$$

(assuming that $\dim(P \cap (c + \mathbb{R}^{d_i})) < d_i$ for all $c \in \mathbb{R}^d$). That is, we first slice P by subspaces parallel to $\mathbb{R}^{d_i}$ and then replace each slice $\Pi(\mu, \nu)$ with $\Pi(\mu, \nu')$. Note that P is a facet of $D_i(P)$ unless $D_i(P) = P$.

Example 1 (case of GL_n): Consider the root space (1) with the functions l_i given by the formula:

$$l_i(x) = \sigma_{i-1}(x) + \sigma_{i+1}(x),$$

where $\sigma_i(x) = \sum_{k=1}^{d_i} x_k^i$ for $i = 1,\ldots, n-1$ and $\sigma_0 = \sigma_n = 0$. It is not hard to show that the Gelfand–Zetlin polytope Q_λ defined above coincides with the polytope

$$[(D_1 \ldots D_{n-1})(D_1 \ldots D_{n-2}) \ldots (D_1)]\,(p),$$

where $p \in \mathbb{R}^d$ is the point $(\lambda_2, \ldots, \lambda_n; \lambda_3, \ldots, \lambda_n; \ldots; \lambda_n)$.

Example 2 (arbitrary reductive groups): Let G be a connected reductive group of semisimple rank n. Let $\alpha_1,\ldots, \alpha_n$ denote simple roots of G, and $s_1,\ldots, s_n$ the corresponding simple reflections. Fix a reduced decomposition $w_0 = s_{i_1} \ldots s_{i_d}$ of the longest element in the Weyl group of G. Let d_i be the number of s_{i_j} in this decomposition such that $i_j = i$. Consider the root space

$$\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \ldots \oplus \mathbb{R}^{d_n} \tag{2}$$

with the functions l_i given by the formula:

$$l_i(x) = \sum_{k \neq i} (\alpha_k, \alpha_i) \sigma_k(x),$$

where (α_k, α_i) is the element of the Cartan matrix of G, that is, $s_i(\alpha_k) = \alpha_k - (\alpha_k, \alpha_i)\alpha_i$. In particular, for $G = GL_n$ and $w_0 = (s_1 \ldots s_{n-1})(s_1 \ldots s_{n-2}) \ldots (s_1)$ we get the root space of Example 1. Define the projection of the root space to the real span of the weight lattice of G by the formula $p(x) = \sigma_1(x)\alpha_1 + \ldots + \sigma_n(x)\alpha_n$.

Theorem 1.2. *For each dominant weight λ of G there exists a point $p_\lambda \in \mathbb{R}^d$ such that the polytope*

$$P := D_{i_1} \ldots D_{i_d}(p_\lambda)$$

yields the Weyl character $\chi(V_\lambda)$ of the irreducible G-module V_λ, namely,

$$\chi(V_\lambda) = \sum_{x \in P \cap \mathbb{Z}^d} e^{p(x)}.$$

Similarly, the face of P defined as $D_{i_l} \dots D_{i_d}(p_\lambda)$ yields the Demazure character corresponding to $w = s_{i_l} \dots s_{i_d}$ and λ for any $l \leq d$. Demazure characters for the other elements of the Weyl group can be represented by unions of faces of P. This can be used to generalize mitosis on Schubert polynomials by Knutson–Miller from $GL(n)$ to other reductive groups.

2. Publications and preprints

[KST] joint with Evgeny Smirnov and Vladlen Timorin, *Schubert calculus and Gelfand-Zetlin polytopes*, Russian Mathematical Surveys, **67** (2012), Issue 4, 685–719

[K1] *Divided difference operators on convex polytopes*, Oberwolfach reports, 21/2012, 5–7

[K2] *Convex Geometry and Representation Theory*, Proceedings of the conference Information Technology and Systems - 2012 (Petrozavodsk, Russia), 450–453

[AKP] joint with Klaus Altmann and Lars Petersen, *Merging divisorial with colored fans*, 30 pages, arXiv:1210.4523 [math.AG], submitted to Michigan Math. J.

[GKT] joint with Pavel Gusev and Vladlen Timorin, *Number of vertices in the Gelfand–Zetlin polytopes*, 11 pages, arXiv:1205.6336 [math.CO], submitted to J. Comb. Theory, Ser. A

[KK] joint with Amalendu Krishna, *Equivariant Cobordism of Flag Varieties and of Symmetric Varieties*, 18 pages, arXiv:1104.1089 [math.AG], submitted to Transformation Groups (last revised in Sept. 2012)

3. Talks

Conference talks

April	Oberwolfach workshop on toric geometry, Oberwolfach, Germany
July	The 5th MSJ-SI Schubert calculus, Osaka, Japan
October	Banff workshop on Lie algebras, torsors and cohomological invariants, Banff, Canada

Seminar talks

February	Seminar “Geometry, Topology and Mathematical physics”, Steklov Institute of Mathematics, Moscow
March	Postnikov seminar “Algebraic topology and its applications”, Moscow State University
May	Seminar “Riemann surfaces, Lie algebras and Mathematical physics” , Independent University of Moscow

4. International collaboration

I work on the following projects with international collaborators.

• "Spherical varieties and polyhedral divisors on wonderful compactifications", joint with Klaus Altmann and Lars Petersen (Freie Universität Berlin)

• "Equivariant cobordism of spherical varieties", joint with Amalendu Krishna (Tata Institute, Mumbai)

• "Newton–Okounkov polytopes of flag varieties and of Bott-Samelson resolutions", joint with Dave Anderson (U. of Washington) and Kiumars Kaveh (U. of Pittsburgh)

5. Teaching

I teach on a regular basis at the Faculty of Mathematics, Higher School of Economics. In 2011-2012, together with Alexander Kolesnikov, I taught a course "Calculus of variations and optimal control":

`http://www.hse.ru/edu/courses/34463194.html`

In Spring 2012, I also taught a course "Mathematical experiments using Mathematica":

`http://www.hse.ru/edu/courses/34469965.html`

In Fall 2012, together with Nina Sakharova I conduct Calculus I problem solving sessions for beginners:

`http://vyshka.math.ru/1213/calculus-1.html`

In 2012-2013, together with Alexander Esterov, Alexander Kolesnikov and Evgeny Smirnov, I run an undergraduate learning seminar "Convex geometry" devoted to various applications of convexity in geometry (including algebraic geometry), combinatorics, analysis, number theory and representation theory:

`http://math.hse.ru/nis-12-vgeom`

In February 2012, I gave a lecture "Enumerative geometry and 3264 conics" at the winter mathematical school for university students organized by the HSE. In May 2012, I gave 3 lectures on topology and intersection theory at the Spring School in Mathematics and Physics for students organized by the Department of Mathematics, HSE:

`http://www.mccme.ru/~valya/presents.html`

I supervise a 4th year student (Diploma "Number of vertices in Gelfand–Zetlin polytopes for 4-step flag varieties"), two 3d year students (course projects "Construction of polytopes via divided difference operators" and "Automorphisms of horospherical varieties"), one 2d year student ("Solvability and unsolvability of equations in explicit form") and two 1st year students ("Polytopes and equations" and "Continued fraction for e").

At the HSE, I coordinate the PhD program in Mathematics. In particular, I was responsible for the admission exams to the regular PhD program as well as to the new "academic" PhD program in Mathematics:

`http://math.hse.ru/post-graduate`

6. Conference organization

Together with Kiumars Kaveh, Evgenia Soprunova, Ivan Soprunov and Vladlen Timorin we organized the international conference "Algebra and Geometry" dedicated to the 65-th anniversary of Askold Khovanskii:

`http://bogomolov-lab.ru/AG2012/Askoldfest2012.htm`

The conference was supported by the Laboratory of algebraic geometry and its applications and Department of Mathematics HSE, Independent University of Moscow, RFBR, Dynasty foundation and the HSE Academic Fund.

Together with Nathan Ilten and Hendrik Süss we submitted a proposal for an Oberwolfach miniworkshop "Algebraic torus actions of non-zero complexity".

7. Comparing the results with the original plans

In the original proposal, I outlined two directions for future research: (1) extending theory of Newton polytopes to spherical varities, and (2) computing explicitly algebraic cobordism rings. I also proposed two concrete problems in these directions: (1.0) Schubert calculus in terms of Gelfand–Zetlin polytopes, and (2.0) explicit formulas for push-forward morphisms in algebraic cobordism in terms of divided difference operators.

Together with Evgeny Smirnov and Vladlen Timorin we solved problem (2.0) in 2010, and published the results in 2012 [KST]. In 2010, I found other concrete problems in directions (1) and (2) due to collaboration with Klaus Altmann and Lars Petersen (extending theory of polyhedral fans to spherical varieties) and with Amalendu Krishna (computing equivariant algebraic cobordism rings of some spherical varities). None of these problems was stated in the original proposal because collaboration started only in 2010. Both projects were successful and the results are contained in preprints [AKP, KK] submitted for publication.

There are few other results in direction (1). In 2011, I realized that some constructions of [KST] led to a general construction of convex-geometric divided difference operators, and in 2012, I studied applications of this construction [K1,K2]. In 2012, together with Pavel Gusev and Vladlen Timorin we studied combinatorics of Gelfand–Zetlin polytopes [GKT].

I have not tried to solve problem (2.0), since I chose instead to work with Amalendu Krishna on equivariant cobordism. However, I plan to return to (modifications of) problem (2.0) since I am very interested in divided difference operators in their various incarnations.