REPORT ON PIERRE DELIGNE FELLOWSHIP, 2010

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During 2010 I have been working on developing a representation theory approach to mirror symmetry and quantum cohomology of homogeneous spaces, associated to Lie groups of classical type.

According to Givental the $\mathfrak{gl}_N$-Whittaker functions, being solutions to the quantum cohomology D-module $QH^*(\mathrm{Fl}_N)$ of the complete flag variety $\mathrm{Fl}_N = GL_N(\mathbb{C})/B$, describe the corresponding equivariant Gromov-Witten invariants of Fl_N. However, the Givental's approach to representation theory description of quantum cohomology of homogeneous spaces is inapplicable to generic incomplete flag variety $\mathrm{Fl}_{m_1,\dots,m_k}$, since no relevant Whittaker model (Toda lattice) associated with an incomplete flag variety was known.

In (2) a generalization of the $\mathfrak{gl}_N$-Whittaker function to the case of the Grassmann variety $\mathrm{Gr}_{m,\,N} = GL_N(\mathbb{C})/P_m$, $1 \leq m < N$ is proposed. Namely, in (2) we define a Toda-type D-module and its solution $\Psi^{(m,N)}_{\lambda_1,\dots,\lambda_N}(x_1,\dots,x_N)$ (referred to as $\mathrm{Gr}_{m,\,N}$-Whittaker function), such that after specialization $x_2 = \ldots = x_N = 0$ the symbols of this D-module reproduce the small quantum cohomology algebra $qH^*(\mathrm{Gr}_{m,\,N})$ due to Astashkevich and Sadov, and Kim. Conjecturally, the constructed generalized Whittaker function describe the equivariant Gromov-Witten invariants of $\mathrm{Gr}_{m,N}$, and in (2) we verify this conjecture in particular case of projective space $\mathbb{P}^{N-1} = \mathrm{Gr}_{1,N}$.

The main result of (2) establishes an explicit relation between the (Archimedean) Langlands duality and mirror symmetry in two-dimensional topological quantum field theories. Namely, in (2) we construct two different representations of the $\mathbb{P}^{N-1}$-Whittaker functions as correlation functions in topological quantum field theories on a two-dimensional disk. In one case the $\mathbb{P}^{N-1}$-Whittaker function is given by a correlation function in a type A equivariant topological sigma model with the target space $\mathbb{P}^{N-1}$. In the other case the same $\mathbb{P}^{N-1}$-Whittaker function appears as a correlation function in a type B equivariant topological Landau-Ginzburg model related with the type A model by mirror symmetry. From this perspective the existence of two mirror dual topological field theory representations of the parabolic Whittaker functions provide a quantum field theory realization of the local Archimedean Langlands duality for Whittaker functions.

In (3) we construct Mellin-Barnes type integral representation of the specialized $\mathrm{Gr}_{m,N}$-Whittaker function, following an original generalization of Whittaker models to incomplete flag manifolds from (2):

$$\Psi^{(m,\,N)}_{\underline{\lambda}}(x,0,\dots,0) \;=\; \int\limits_{\mathcal{C}} d\underline{\gamma}\; e^{\,-\frac{x}{\hbar}\sum\limits_{i=1}^{m}\gamma_i}\; \frac{\prod\limits_{i=1}^{m}\prod\limits_{j=1}^{N}\hbar^{\frac{\gamma_i-\lambda_j}{\hbar}}\;\Gamma\left(\frac{\gamma_i-\lambda_j}{\hbar}\right)}{\prod\limits_{\substack{i,k=1\\ k\neq i}}^{m}\hbar^{\frac{\gamma_i-\gamma_k}{\hbar}}\;\Gamma\left(\frac{\gamma_i-\gamma_k}{\hbar}\right)}\,.$$

with $\underline{\gamma} = (\gamma_1,\dots,\gamma_m)$ and $\underline{\lambda} = (\lambda_1,\dots,\lambda_N)$. Our derivation involves a generalization of the Gelfand-Zetlin realization to infinite-dimensional $U(\mathfrak{gl}_N)$-modules introduced Gerasimov, Kharchev, and Lebedev. Besides, our integral representation verifies the conjectural

integral formula by Hori and Vafa, although our solution to the quantum cohomology D-module has a different asymptotic behavior.

The paper (1) is devoted to construction of elementary integral intertwining operators for quantum Toda chains associated to (affine) root systems of classical types. The constructed operators turn out to intertwine Toda chains corresponding to root systems of different types and ranks. In particular, compositions of such intertwiners produce recursive operators and Baxter's Q-operators, whose representation theory description was given previously in our papers. We also conjecture explicit form of $\mathcal{Q}$-operators for closed Toda chains corresponding to Lie algebras of types B_∞, C_∞, D_∞, affine Lie algebras of types $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $D_n^{(2)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$ and the affine analogs of BC_n and Inozemtsev Toda chains.

§1. Publications

(1) *Quantum Toda chains intertwined* (joint with A. Gerasimov and D. Lebedev), St. Petersburg Math. J. (2010).

(2) *Parabolic Whittaker functions and Topological field theories I* (joint with A. Gerasimov and D. Lebedev), Preprint `[hep-th/1002.2622]`, 2010, 46 pages.

(3) *On parabolic Whittaker functions*, Preprint `[math.AG/1011.4250]`, 2010, 15 pages.

§2. Teaching

In Autumn semester 2010 I have given a course on spinors to the 3-d and the 4-th year students of Moscow Institute of Physics and Technology (MIPT) at ITEP. The course is an introduction to supersymmetry, and will be continued in the next semester.

Besides, together with A. Gerasimov and D. Lebedev, I organize a seminar on quantum field theory for the MIPT students at ITEP.