Final report on Pierre Deligne Fellowship

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During the last three years 2009-2012 I have been working on a big project in collaboration with A.Gerasimov and D.Lebedev, which is devoted to novel and transformative interactions of Quantum Physics (topological quantum field theory, quantum integrable systems and quantum statistical mechanics) with Number theory and Automorphic forms on the base of Representation theory. The central part of this project is concerned with an innovative and effective approach to the Langlands program, developed in our previous work.

Recall, that in the case of p-adic field K the local Langlands correspondence identifies the two analytic objects, the L-functions; the first one is constructed by a finite-dimensional representation of the Galois group (more precisely, of its extension, the Weil-Deligne group W_K), $W_K \to G^\vee$, and the other L-function is defined by an infinite-dimensional representation of a finite-dimensional reductive group G. Over the non-Archimedean field K there is also another interpretation of the local Langlands correspondence in terms of special functions on G given by matrix elements in infinite-dimensional representations of G; one possibility for such a special function is the G-Whittaker function. Namely, in his letter to Godement, Langlands conjectured an explicit formula for the p-adic Whittaker function, which identifies the G-Whittaker function with a character of the finite-dimensional representation of the (complex) dual group $G^\vee$. This formula was proved for $GL(N)$ by Shahidi, and then for arbitrary semisimple G by Casselman and Shalika. In the case of Archimedean field K the Langlands correspondence is less studied, and its description is much less transparent, particularly, due to a highly nontrivial nature of the Archimedean Weil-Deligne group. Besides, the Archimedean analog of the Langlands-Shahidi formula was not known until very recently.

In [9] a q-version of the Langlands-Shahidi formula was discovered. Namely, by analogy with the case of a real group, the q-deformed Whittaker function can be defined as a solution to the associated q-Toda chain; in particular, it can be identified with the specialization of the Macdonald polynomials. [9] shows that the q-deformed $GL(N)$-Whittaker function, which can be identified with a matrix element in an infinite-dimensional representation of the quantum group $U_q(\mathfrak{gl}_N)$, coincides with the character of finite-dimensional Demazure module of the affine Lie algebra $\widehat{\mathfrak{gl}}_N$. This identity should be viewed as a q-version of the Langlands-Shahidi formula, and, more generally, it is a manifestation of the local Langlands correspondence. Namely, after specialization of q to the power of p the q-version of the Langlands-Shahidi formula reduces to the original, p-adic, Langlands-Shahidi formula. Besides, [4] implies that under the limit $q \to 1$ the Gelfand-Zetlin explicit representation of the q-deformed $GL(N)$-Whittaker function from [9] results into the Givental's stationary phase integral representation of the $GL(N, \mathbb{R})$-Whittaker function. Thus the q-version of the Langlands-Shahidi formula, proposed in [9], interpolates between the Archimedean and non-Archimedean cases, and allows for a novel method in the study of the local Langlands correspondence uniformly for the both types of base local fields. This provides an essential progress in the study of the Archimedean Langlands correspondence.

The works [6]-[8] pioneered a geometric construction of the Archimedean Langlands correspondence in terms of the topological quantum field theory (TQFT). Particularly, in [8] a new integral representation of the Euler's Gamma-function in terms of the functional integral was given, which allowed to identify the classical Gamma-function with the $S^1 \times U(1)$-equivariant volume of the space of holomorphic $\mathbb{C}$-valued functions on two-dimensional disk $D = \{z \in \mathbb{C} : \, |z| \leq 1\}$. Moreover, in [8] the Archimedean L-functions (given by a product of Gamma-functions) were identified with certain correlation functions in type A TQFT associated with the space of maps $D \to V$ into the complex finite-dimensional representation of $GL(N)$; on the other hand [7], they coincide with certain correlation functions in type B Landau-Ginzburg model on a complex space with a superpotential $\mathcal{W}$. The involved topological quantum field theories of types A and B turned out to be mirror symmetric to each other, and thus [7], [8] identify the Mirror Symmetry with the Archimedean Langlands correspondence. Besides, [6] discovered an Archimedean analog of the Langlands-Shintani formula for the (parabolic) $GL(N)$-Whittaker functions in the case of the projective space $\mathbb{P}^{N-1}$ in terms of the pair of mirror symmetric topological quantum field theories on two-dimensional disk D. Thus the results of [6]-[8] propose a new model for the geometric representation theory over Archimedean places, effectively applying methods and constructions from string theory and quantum integrable systems.

In [6], [5], [3] a new class of special functions on $GL(N)$ was discovered, the parabolic Whittaker functions, associated with Grassmann varieties $GL(N)/P_m = \mathrm{Gr}_{m;\,N}$, extending the classical $GL(N)$-Whittaker functions, associated with complete flag variety $GL(N)/B$. Parabolic $GL(N)$-Whittaker functions are expected to describe the $S^1 \times U(N)$-equivariant Gromov-Witten invariants of the associated (partial) flag varieties. In particular, in [5] and [3] in the case of Grassmannian the corresponding parabolic $GL(N)$-Whittaker functions were identified with certain matrix elements in a principal series representation of $GL(N, \mathbb{R})$, thus providing a partial solution to an old problem of defining the Whittaker functions associated with general flag varieties G/P. Besides, a pair of the dual integral representations, the Mellin-Barnes and the stationary phase, of the $\mathrm{Gr}_{m;\,N}$-Whittaker functions was given. Remarkably, the Mellin-Barnes integral representation of the $\mathrm{Gr}_{m;\,N}$-Whittaker function [5] reproduces the vortex limit of the Nekrasov's instanton counting function, and the stationary phase integral from [3] coincides with the Batyrev-Ciocan-Fontanine-Kim-van Straten integrals for the generating functions of quantum cohomology of Grassmannians. Evidently, [6], [5] and [3] enrich modern representation theory and harmonic analysis with constructions and methods from string theory, and allow for an essential progress in these areas.

Another key tool in our approach to Langlands program is the Baxter $\mathcal{Q}$-operator formalism, which reproduces the Hecke algebra symmetry of the underlying quantum integrable system (e.g. Toda chain). Also Baxter operators are closely connected to the recursive integral operators and thus they provide explicit integral representations of matrix elements on a group G. [1] proposed basic ingredients for the Baxter $\mathcal{Q}$-operator formalism for the Macdonald polynomials and Jack polynomials of type A_n and for the q-deformed $GL(N)$-Whittaker functions; particularly, the (q,t)-deformations of the Archimedean L-functions were introduced. The results of [2] and [10] show that our approach can be readily extended to other classical series, other than A_n. In particular, [2] contains explicit expressions for the superpotentials in type B TQFT on disk D reproducing the $SO(N)$- and $Sp(2n)$-Whittaker fucntions in the sense of [6]. Besides, [2] proposes stationary phase integral representations of G-Whittaker functions for all classical groups G, extending the Givental's integral formula for $G = GL(N)$.

Summarizing, my results are enriching the algebro-geometric approach to the Langlands correspondence by novel analytic and geometric structures (maps $D \to V$ and the LG model) and symmetries (Mirror Symmetry, quantum integrability, quantum inverse scattering method), discovered recently in my research. I hope that they will transform the landscape in representation theory and automorphic forms, and will ensure an advanced progress in the Langlands program in the closest future.

Finally, I appreciate Pierre Deligne for the support of my research, and also for his interest, attention and feedback at our annual Christmas meetings at the Independent University of Moscow. I also would like to express my admiration with his efforts, patience and purposefulness in contributing to the support of young mathematicians in Russia.

References

[1] *On Baxter operator formalism for Macdonald polynomials*, (with A. Gerasimov and D. Lebedev), Preprint [math.AG/1204.0926], 2012, 21 pages.

[2] *New integral representations of Whittaker functions for classical groups*, (with A. Gerasimov and D. Lebedev), Rus. Math. Surveys, 67:1 (2012) 3-96.

[3] *On parabolic Whittaker functions II*, Cent. Eur. J. Math. 10:2 (2012) 543-558.

[4] *On a classical limit of q-deformed Whittaker functions*, (with A. Gerasimov and D. Lebedev), Lett. Math. Phys., 100 (2012) 279-290.

[5] *On parabolic Whittaker functions*, Lett. Math. Phys., 101 (2012) 289-304.

[6] *Parabolic Whittaker functions and topological field theories I*, (with A. Gerasimov and D. Lebedev), Commun. Number Theory and Physics, 5 (2011) 135-202.

[7] *Archimedean L-factors and topological field theories II*, (with A. Gerasimov and D. Lebedev), Commun. Number Theory and Physics, 5 (2011) 101-134.

[8] *Archimedean L-factors and topological field theories*, (with A. Gerasimov and D. Lebedev), Commun. Number Theory and Physics, 5 (2011) 57-101.

[9] *On q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function III*, (with A. Gerasimov and D. Lebedev), Lett. Math. Phys. 97 (2011), 1-24.

[10] *Quantum Toda chains intertwined*, (with A. Gerasimov and D. Lebedev), Special volume on the occasion of 75-th Anniversary of L. D. Faddeev, St. Petersburg Math. J. 22 (2011), 411-435.

Seminars and Conferences

1. Oberseminar *Orthogonal polynomials, Whittaker functions and the local Langlands correspondence*, at Max-Planck-Institut für Mathematik in Bonn, September 2012

2. Seminar *Baxter operators, automorphic L-functions and topological field theories*, at University of Nottingham, UK, August 2012

3. Talk *Parabolic Whittaker functions and quantum cohomology of homogeneous spaces*, at "Classical and Quantum Integrable Systems", Protvino, Russia, January 2011

4. Plenary talk *Whittaker functions and Topological field theories* at "Whittaker funcitons, Crystals and Quantum groups", Banff, Canada, June 2010

5. Oberseminar *q-deformed Whittaker functions and Demazure modules*, Max-Planck-Institut für Mathematik in Bonn, January 2010

Teaching

In Autumn semester 2010 I have given a course on spinors to the 3-d and the 4-th year students of Moscow Institute of Physics and Technology (MIPT) at ITEP. This course is an introduction to supersymmetry, focused on the representation-theoretic aspects of the subject. In Spring 2011 I gave a course on orbit method and representation theory of noncompact groups to the 3-d and the 4-th year students of Moscow Institute of Physics and Technology (MIPT) at ITEP. This course continues a course on spinors, that I gave in 2010.

Besides, I co-organized (with A. Gerasimov and D. Lebedev) a seminar on mathematical apsects of quantum field theory for the MIPT students at ITEP.