# Motivic integration and knot homology

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The main purpose of the research is the study of motivic integrals and their relation to invariants of plane curve singularities, knots and links, in particular, to the invariant related to knot (and link) homology theories. It is supposed to enlarge the relation between the motivic Poincare series of plane curve singularities and the Heegard-Floer homology of corresponding algebraic knots.

## 1 Framework

### 1.1 Motivic integration

Motivic integration was introduced in 1995 by M. Kontsevich, and later developed by J. Denef and F. Loeser in a series of articles (see, e.g. [4]).

The idea of motivic integration has several sources. First, it is a deep generalization of the integration with respect to the Euler characteristic ([19], [11]). Second, it is tightly related to the p-adic integration arising in number theory and arithmetic geometry. Third, the idea of motivic integration is close to the open string theory in mathematical physics, as integration is taken over the space of (formal) germs of maps from complex line to a given target space. Since complex line is 2-dimensional, its image may be considered as a (singular) worksheet of an open string.

The novelty of the motivic integration is the set of values of the motivic measure, which is is a certain completion of the localization of the Grothendieck ring of varieties by the class of the affine line. The motivic measure can be defined on different infinite-dimensional functional spaces which can be approximated by the finite-dimensional jet spaces. The algebra of measurable sets is, roughly speaking, generated by the sets defined by a finite collection of semi-algebraic equations on jets.

One can define, for example, a motivic measure on the set of arcs on a given variety (formal maps from a complex line to variety). For arc spaces, the main tool of study is the change of variables formula ([4]) that provides a possibility of replacement of a variety by its resolution. A remarkable example of usage of the change of variables formula is the proof of the Batyrev's conjecture ([1]): birationally equivalent Calabi-Yau smooth varieties have equal Hodge numbers.

### 1.2 Poincare series and Alexander polynomial

In a series of articles (e.g. [2]), A. Campillo, F. Delgado and S. Gusein-Zade proved that the Alexander polynomial of the link of a plane curve singularity is related to the generating function arising in a purely algebraic setup.

Let $C = \cup_{i=1}^{r} C_i$ be a germ of a plane curve, let $\gamma_i : (\mathbb{C}, 0) \to (C_i, 0)$ be uniformizations of its components. If $f \in \mathcal{O} = \mathcal{O}_{\mathbb{C}^2,0}$ is a germ of a function on $(\mathbb{C}^2, 0)$, we define $v_i(f) = Ord_0 f(\gamma_i(t))$, and the Poincare series of the curve $C$ is defined ([2]) as the integral with respect to the Euler characteristic

$$P^C(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}} t_1^{v_1} \cdot \ldots \cdot t_r{}^{v_r} d\chi, \qquad (1)$$

where $\mathbb{P}\mathcal{O}$ denotes the projectivization of $\mathcal{O}$ as a vector space.

Let $\Delta^C(t_1, \ldots, t_n)$ denote the Alexander polynomial of the link of $C$. The theorem of Campillo, Delgado and Gusein-Zade says that if $r = 1$, then $(1-t)P^C(t) = \Delta^C(t)$, and if $r > 1$, then $P^C(t_1, \ldots, t_r) = \Delta^C(t_1, \ldots, t_r)$.

In [3] there was proposed the following generalization of the Poincare series. One can naturally define a motivic measure on the projectivization of the space of functions, and consider the following motivic integral, generalizing (1):

$$P_g^C(t_1, \ldots, t_r) = \int_{\mathbb{P}\mathcal{O}} t_1^{v_1} \cdot \ldots \cdot t_r{}^{v_r} d\mu. \tag{2}$$

If $r = 1$, one can deduce $P_g(t)$ from $P(t)$. If $r$ is greater than 1, the situation becomes more complicated: the motivic Poincare series is still determined by the ordinary one, but the explicit algorithm of its computation is unclear.

## 1.3 Heegard-Floer homology

In a series of articles (e.g. [14]),[15]) P. Ozsvath and Z. Szabo constructed the so-called Heegard-Floer knot homology theory. It assigns to any knot or link in a 3-dimensional manifold a filtered chain complex, such that the Euler characteristic of the adjoined (bi)graded complex coincides with the Alexander polynomial. This homology theory has several remarkable topological properties: the maximal index of the filtration level with non-zero homology coincides with the knot genus ([16]), that is, the minimal genus of a Seifert surface for the knot in $S^3$. Furthermore, a knot is fibered if and only if this top homology group is one-dimensional ([13]).

Later a general combinatorial model for the Heegard-Floer complex was developed ([12]), but its size seems to make the structure of the homology unclear: for example, for a trefoil knot it has 5!=120 generators.

For all algebraic knots (for example, torus ones) Ozsvath and Szabo managed ([17]) to calculate explicitly the Heegard-Floer homology, namely, it can be reconstructed by a purely combinatorial procedure from the Alexander polynomial.

## 1.4 Unification of knot homology theories

Several knot homology theories different from Heegard-Floer homology were constructed.

M. Khovanov ([9]) constructed a homology theory categorifying the Jones polynomial. Later Khovanov and Rozansky gave a unified construction ([10]) of the homology theories categorifying $sl(N)$ colored Jones polynomials, and another homology theory categorifying the HOMFLY polynomial.

Although the constructions of Khovanov and Rozansky are combinatorial and use only knot diagrams, the explicit Poincare polynomials of the corresponding homology groups even for torus knots are known only in some particular cases.

To get all these theories together, Dunfield, Gukov and Rasmussen ([5]) conjectured that they are parts, or specializations of a following unified picture. Namely, for a given knot $K$ they conjectured existence of a triply-graded knot homology theory $\mathcal{H}_{i,j,k}(K)$ with the following properties:

- **Euler characteristic**. Consider the Poincare polynomial $\mathcal{P}(K)(a, q, t) = \sum a^i q^j t^k \dim \mathcal{H}_{i,j,k}$. Then $\mathcal{P}(K)(a, q, -1)$ equals the value of the HOMFLY polynomial of the knot $K$.

- **Differentials.** There exist a set of anti-commuting differentials $d_j$ for $j \in \mathbb{Z}$ acting in $\mathcal{H}_*(K)$. For $N > 0$, $d_N$ has triple degree $(-2, 2N, -1)$, $d_0$ has degree $(-2, 0, -3)$ and for $N < 0$ $d_N$ has degree $(-2, 2N, -1 + 2N)$.

- **Symmetry.** There exists a natural involution $\phi$ such that $\phi d_N = d_{-N}\phi$ for all $N \in \mathbb{Z}$.

For $N \geq 0$, the homology of $d_N$ is supposed to be tightly related to the $sl(N)$ Khovanov-Rozansky homology. Namely, let $\mathcal{H}^N_{p,k}(K) = \oplus_{iN+j=p}\mathcal{H}_{i,j,k}(K)$. It was conjectured in [5] that there exists a homology theory with above properties such that for all $N > 0$ the homology of $(\mathcal{H}^N_*(K), d_N)$ is isomorphic to the $sl(N)$ Khovanov-Rozansky homology. For $N = 0$, $(\mathcal{H}^0_*(K), d_0)$ is isomorphic to the Heegard-Floer knot homology.

## 2 Author's results related to the project

In [23] and [22] a set of functional equations for motivic integrals was developed and studied. These integrals have no explicit closed formula, while the change of variables formula provides natural functional equations for them. These equations can be used for a recursive calculation of the coefficients in the power series decomposition of the integral with respect to the parameters. The resulting equations can be used in proof of some unexpected properties on an integral, for example, its symmetry under an inversion of the parameters ([23]).

One of the integrals related to the Milnor number over the space of arcs on the complex plane is studied in [23] and [22]. Such an integral naturally arises in the following context. A function on $(\mathbb{C}^2, 0)$ defines a possibly reducible germ of a curve. One can parametrize each irreducible component of it and hence consider a map from the set of functions to the set of unordered tuples of arcs modulo automorphisms of $(\mathbb{C}, 0)$. It turns out that the motivic measures on these two functional spaces are related by some factor ([24]). This factor has a clear geometric meaning - it is expressed through the number of self-intersections of a generic deformation on a curve, which for irreducible curves is half of the Milnor number.

The results of [25] are the most important for the current project. The explicit algorithm for calculation of the motivic Poincare series is presented in terms of the geometry of the embedded resolution.

The reduced motivic Poincare series $\overline{P}^C_g$ is defined as

$$\overline{P}^C_g(t_1, \ldots, t_r) = (1 - qt_1) \cdot \ldots \cdot (1 - qt_r) \cdot P^C_g(t_1, \ldots, t_r).$$

In [25] it is proved that the reduced motivic Poincare series satisfies the following properties.

1. **Polynomiality.** $\overline{P}_g(t_1, \ldots, t_n; q)$ is a polynomial in $t_1, \ldots, t_n$ and $q$. A bound for its degree in $t_1, \ldots, t_n$ is given.

2. **Reduction to the Alexander polynomial.** If $n = 1$, then $\overline{P}_g(t; q = 1) = \Delta(t)$, where $\Delta$ denotes the Alexander polynomial of the link of the corresponding plane curve singularity. If $n > 1$, then $\overline{P}_g(t_1, \ldots, t_n; q = 1) = \Delta(t_1, \ldots, t_n) \cdot \prod_{i=1}^n (1 - t_i)$.

3. **Forgetting components.** Let $C$ be a curve with $n$ components, and $C_1$ be an irreducible curve. Then $\overline{P}^{C \cup C_1}_g(t_1, \ldots, t_n, t_{n+1} = 1) = (1 - q)\overline{P}^C_g(t_1, \ldots, t_n)$. If $C$ has only one component, then $\overline{P}^C_g(t = 1) = 1$.

4. **Symmetry.** Let $\mu_\alpha$ be the Milnor number of $C_\alpha$, $(C_\alpha \circ C_\beta)$ is the intersection index of $C_\alpha \circ C_\beta$, $\mu(C)$ is the Milnor number of $C$. Let

$$l_\alpha = \mu_\alpha + \sum_{\beta \neq \alpha} (C_\alpha \circ C_\beta), \quad \delta(C) = (\mu(C) + r - 1)/2.$$

It is known that the Alexander polynomial is symmetric in the sense that

$$\Delta(t_1^{-1}, \ldots, t_n^{-1}) = \prod t_\alpha^{-l_\alpha} \cdot \Delta(t_1, \ldots, t_n).$$

A generalization of this identity is proved. Namely,

$$\overline{P}_g(\frac{1}{qt_1},\ldots,\frac{1}{qt_r}) = q^{-\delta(C)}\prod_{\alpha} t_{\alpha}^{-l_{\alpha}}\cdot\overline{P}_g(t_1,\ldots,t_r).$$

5. **Relation to the knot homology.** For irreducible curves it is proved that $\overline{P}_g(t)$ can be related by a simple procedure with the Poincare polynomial of the Heegard-Floer knot homologies constructed by P. Ozsvath and Z. Szabo.

# 3 Research plan

The project is devoted to the study of the conjectural triply graded homology from [5] for algebraic knots, the algebraic and combinatorial structure of the latter one and its relation to the motivic integration.

The simplest set of examples are torus knots. The expressions for the Alexander polynomials of torus knots are well known, it determines Heegard-Floer homology, but the homology structure is more involved and less understood. In the Heegard-Floer theory, clear combinatorial construction of the complex quasi-isomorphic to the Heegard-Floer complex is still missing for generic torus knots. Furthermore, the uniform description of the Khovanov homology for torus knots is also missing, and the current research is supposed to fill these gaps.

In [5] it has been remarked that there is a natural limit of the triply graded homology of $(n,m)$ torus knots at $m\to\infty$. It turns out that the limit homology of $(n,\infty)$ knots is isomorphic as a vector space to the polynomial algebra of $n-1$ even generators $x_2,\ldots,x_n$ and $n-1$ odd generators $\xi_2,\ldots,\xi_n$. Their degrees are equal to

$$\deg(x_k)=(0,2k,2k-2), \deg(\xi_k)=(2,2k-2,2k-1).$$

The homology of a $(n,m)$ knot is a subcomplex of the homology of $(n,\infty)$ knot, it inherits the "level structure" from the $(n,\infty)$ homology where the level of a monomial is its degree in odd variables. It turns out that the dimension of the level $k$ subspace in the homology of $(m,n)$ knot has a clear combinatorial description. For example, the dimension of the level 0 in the homology of $(n,n+1)$ torus knot is equal to $n$th Catalan number. More precisely, the arising combinatorial structures are related to the $(q,t)$-analogues of Catalan numbers constructed in a series of articles of M. Haiman, A. Garsia and J. Haglund (e.g [8],[6],[7]) in terms of the Macdonald polynomials. The structure of the operators acting in the superhomology is similar to the $A_\infty$ structure in Heegard-Floer homology and spectral sequences related to it.

The research will be focused on the following problems:

1. To explain the relation of the motivic Poincare series for reducible plane curve singularities with the Heegard-Floer homology of algebraic links.

2. To get an algebraic description of the triply graded homology of torus knots in terms of the generators $x_k,\xi_k$ and derive a model for the Heegard-Floer homology of torus knots.

3. To extend the above combinatorial construction to iterated torus knots and algebraic knots.

4. To relate the combinatorial constructions of triply graded homology with motivic integrals over appropriate functional spaces.

5. To relate the combinatorial construction of Heegard-Floer homology for algebraic knots with the topological Heegard-Floer complex.

# References


[1] V. Batyrev. Birational Calabi–Yau n-folds have equal Betti numbers. New trends in algebraic geometry (Warwick, 1996), 1–11, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, 1999.

[2] A. Campillo, F. Delgado, S. M. Gusein-Zade. The Alexander polynomial of a plane curve singularity via the ring of functions on it. Duke Math J. **117** (2003), no. 1, 125–156.

[3] A. Campillo, F. Delgado, S. M. Gusein-Zade. Multi-index filtrations and motivic Poincaré series. Monatshefte *für* Mathematik. **150** (2007), no.3, 193-210.

[4] J. Denef, F. Loeser. Germs of arcs on singular algebraic varieties and motivic integration. Inventiones Math. **135** (1999), no.1, 201–232.

[5] N. Dunfield, S. Gukov, J. Rasmussen. The Superpolynomial for Knot Homologies. J. Experimental Math. **15** (2006), 129-159

[6] A. M. Garsia, M. Haiman. A remarkable $q, t$-Catalan sequence and $q$-Lagrange inversion. J. Algebraic Combin. 5 (1996), no. 3, 191-244.

[7] M. Haiman. $t, q$-Catalan numbers and the Hilbert scheme. Discrete Math. 193 (1998), 201-224.

[8] J. Haglund. The $q, t$-Catalan Numbers and the Space of Diagonal Harmonics with an Appendix on the Combinatorics of Macdonald Polynomials. AMS University Lecture Series Vol.41.

[9] M. Khovanov. A categorification of the Jones polynomial. Duke Math. J. **101** (2000), 359–426.

[10] M. Khovanov, L. Rozansky. Matrix factorizations and link homology. arXiv:math/0401268

[11] R. MacPherson. Chern classes for singular algebraic varieties. Annals of Mathematics **100** (1974), 423–432.

[12] C. Manolescu, P. Ozsvath, Z. Szabo, D. Thurston. On combinatorial link Floer homology. Geom. Topol. **11** (2007), 2339–2412.

[13] Y. Ni. Knot Floer homology detects fibered knots. Invent. Math. **170** (2007), no. 3, 577–608.

[14] P. Ozsvath, Z. Szabo. Holomorphic discs and knot invariants. Adv. Math. **186**(1), 2004, 58–116.

[15] P. Ozsvath, Z. Szabo. Holomorphic discs and link invariants. arXiv:math/0512286

[16] P. Ozsvath, Z. Szabo. Holomorphic discs and genus bounds. Geometry and topology **8** (2004), 311–334.

[17] P. Ozsvath, Z. Szabo. On knot Floer homology and lens space surgeries. Topology **44** (2005), no. 6, 1281–1300.

[18] J. Rasmussen. Knot polynomials and knot homologies, in *Geometry and Topology of Manifolds*, Boden et. al. eds., Fields Institute Communications **47** (2005), 261–280.

[19] O. Viro. Some integral calculus based on Euler characteristic. Topology and Geometry – Rohlin seminar. Lect. Notes Math. **1346**, Berlin, Heidelberg, New York: Springer (1988), 127–138.

[20] M. Yamamoto. Classification of isolated algebraic singularities by their Alexander polynomials. Topology **23** (1984), 277–287.

### Author's papers related to the project

[21] On divisorial filtrations on sheaves. Mathematical notes, 79 (2006), no. 6, 825–837.

[22] Motivic integration and functional equations. Algebra and Analysis
(Journal of the St. Petersburg Mathematical Society), 19 (2007), no. 4, 92–112.

[23] Symmetries of some motivic integrals.
Russian Mathematical Surveys, 63 (2008), no.4 (382), 179–180.

[24] E. Gorsky. On the motivic measure on the space of functions. Revista Matematica Complutense, **20**(2007), no. 2, 507–521.

[25] E. Gorsky. Combinatorial computation of the motivic Poincare series. arXiv:0807.0491

[26] E. Gorsky. Seifert cohomology of trees. arXiv:0901.1298