Research statement

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My recent research is focused on two different fields of algebraic geometry. First, I would like to extend the theory of Newton polytopes to arbitrary spherical varieties. My results in this direction include explicit combinatorial formulas for the Euler characteristic of complete intersections in arbitrary complex reductive groups [14, 15] generalizing the results of D.Bernstein, Khovanskii and Kouchnirenko [20] (all complete intersections in a complex torus) and of Kazarnovskii [17] (zero dimensional complete intersections in any reductive group).

Second, I study the theory of algebraic cobordism and is especially interested in examples where the algebraic cobordism ring of a variety can be described explicitly. Together with Jens Hornbostel, we established Schubert calculus for Bott-Samelson resolutions in the algebraic cobordism rings of complete flag varieties [13] extending results of Bressler–Evens [3] for complex cobordism to the algebro-geometric setting.

My current work is on a relation between the classical Schubert calculus on flag varieties and combinatorics of Gelfand–Zetlin polytopes [16]. Together with Evgeny Smirnov and Vladlen Timorin, we have recently found an interpretation of Schubert calculus in terms of the Pukhlikov-Khovanskii ring associated with the Gelfand–Zetlin polytope.

Below is a detailed description of my results and future plans.

Past research

Euler characteristic of complete intersections. Let $\pi : G \to GL(V)$ be a faithful representation of a connected complex reductive group G, and H_π a generic hyperplane section of G for this embedding. I will now explain how to find the (topological) Euler characteristic of H_π (or more generally, of a complete intersection $H_{\pi_1} \cap \ldots \cap H_{\pi_m}$ for m different representations) in terms of the weight polytope of π. My first step was to construct non-compact *Chern classes* $S_1, \ldots, S_n$ of G (here n denotes the dimension of G) and use them to prove a non-compact adjunction formula for the Euler characteristic $\chi(H_{\pi_1} \cap \ldots \cap H_{\pi_m})$. The Chern classes are elements in the *ring of conditions* of G and can be defined similarly to the classical Chern classes as degeneracy loci of special vector fields on G (namely, one should consider only sums of left- and right-invariant vector fields on G) [14, Section 3.2]. More generally, I constructed Chern classes of arbitrary *spherical homogeneous spaces* [14, Section 5]. These Chern classes turned out to be interesting on their own, in particular, their equivariant versions were recently studied in [5, 6].

Denote by k the rank of G. I proved that the higher Chern classes $S_{n-k+1}, \ldots, S_n$ vanish [14, Lemma 3.8]. The adjunction formula for a hypersurface looks as follows (see [14, Theorem 1.1] for complete intersections of arbitrary dimension):

$$\chi(H_\pi) = (-1)^{n-1}(H_\pi^n - S_1 H_\pi^{n-1} + \ldots + S_{n-k}).$$

In particular, in the torus case all Chern classes vanish (since $n = k$) and we get that the Euler characteristic is equal up to a sign to the self-intersection index of H_π (the latter identity had been for some time expected to hold in the reductive case as well until this was disproved by Kaveh [18]).

In the reductive case, we have to compute the intersection indices S_iH^{n-i} for all $i \leq (n-k)$. To do this it is useful to consider a *regular compactification* X_π of G (an analog of a smooth toric variety) associated with the representation π. Such a compactification can be constructed as the closure of $\mathbb{P}(\pi(G))$ in the projective space $\mathbb{P}(\mathrm{End}(V))$ and is naturally endowed with an action of the doubled group $G \times G$ (as in the torus case it is enough to consider only those representations π for which X_π is regular, in particular, smooth). I refined the algorithm of De Concini–Procesi [9] (originally developed to compute the intersection indices of divisors on wonderful compactifications and then extended by Bifet to regular compactifications [2]) so that it produced the following explicit formula for $S_iH_\pi^{n-i}$ in terms of the weight polytope P_π of π:

$$S_iH_\pi^{n-i} = (n-i)! \int\limits_{P_\pi \cap \mathcal{D}} F_i(x)dx,$$

where $\mathcal{D}$ is a fundamental Weyl chamber and $F_i(x)$ is an explicitly defined polynomial function of degree $n-k-i$ on $\mathbb{R}^k$ that only depends on the group itself and not on π. Note that the usual Chern classes of the compactification X_π can be expressed in terms of S_i so this formula in particular allows to compute the intersection indices of the Chern classes with divisors in regular compactifications.

One way to characterize the polynomial F_i is to consider the Picard group of the double flag variety $G/B \times G/B$ (note that the dimension of $G/B \times G/B$ is precisely $(n-k)$). The Picard group can be identified with $\mathbb{Z}^k \oplus \mathbb{Z}^k$ (if G is semisimple) so that each integer point $(x, y) \in \mathbb{R}^k \oplus \mathbb{R}^k$ defines a divisor $D_{(x,y)}$. Then

$$F_i(x) = \frac{c_iD^{n-k-i}(x, x)}{(n-k-i)!}, \qquad (1)$$

where $c_i = c_i(G/B \times G/B)$ is the usual i-th Chern class of the tangent bundle of the double flag variety $G/B \times G/B$ (see [15, Section 4] for an explicit formula for F_i). In particular, for $i = 0$ (we set $S_0 = G$) the formula for $S_iH_\pi^{n-i}$ reduces to the Brion-Kazarnovskii formula [17, 4]. I use exactly this interpretation of F_i in my proof because the De Concini–Procesi algorithm reduces all computations to the closed $G \times G$–orbits of X_π and the latter are isomorphic to the double flag variety. In particular, this approach gives a new proof of the Brion–Kazarnovskii formula.

The next natural task is to find the Hodge–Deligne numbers of complete intersections in reductive groups. In the torus case many results in this direction were obtained by Danilov and Khovanskii [10] but the reductive case (as the computation of the Euler characteristic already shows) presents many additional challenges and calls for new tools and ingredients.

Schubert calculus in algebraic cobordism (joint with Jens Hornbostel). Algebraic cobordism theory Ω has been recently developed in work of Levine, Morel and Pandharipande

[23, 24]. The theory conveys the classical complex cobordism theory MU to the algebro-geometric setting. In particular, algebraic cobordism $\Omega^n(X)$ for a smooth algebraic variety X allows a presentation with generators being projective morphisms $Y \to X$ of relative codimension $n(:= \dim(X) - \dim(Y))$ and relations given by a *double point relation* (a refinement of the naive cobordism relation) introduced by Levine and Pandharipande [24] .

Let X be the complete flag variety for a split reductive group over a field of zero characteristic. We have described the algebraic cobordism ring $\Omega^*(X)$. First, we showed that there is a natural basis given by the cobordism classes of Bott–Samelson resolutions of Schubert cycles. Second, we expressed these classes in terms of the first Chern classes of line bundles on X using generalized Demazure or divided difference operators ([3], Theorem 3.2). This extends to algebraic cobordism analogous results for the Chow ring of X [1, 11] and for the complex cobordism ring $MU^*(X_{\mathbb{C}})$ [3]. Note that in [3] Bressler–Evens use homotopy theory methods (which can not be extended to algebraic cobordism), while our methods are purely algebro-geometric. We also proved a cobordism version of the Chevalley formula [13, Proposition 4.3] which allowed us to give an efficient algorithm for multiplying two Bott–Samelson classes [13, Section 5].

One of the main tools for us was Vishik–Quillen formula [26] for the push-forward (also called Gysin map) in algebraic cobordism in the case of projective line fibrations. In this case, we found a new geometric proof of this formula using the double point relation [13, proof of Proposition 2.1].

Work in progress and future plans

Schubert calculus and Gelfand-Zetlin polytopes. One of my future goals is to describe the cohomology rings of regular compactifications of reductive groups by generators and relations using Newton polytopes. This is motivated by the following unpublished result of Pukhlikov and Khovanskii. To each convex polytope P they assigned a graded commutative ring R_P using the volume polynomial of P. In the case where P is simple, the elements of R_P can be identified with linear combinations of faces of P modulo some explicit relations. If P is an integrally simple lattice polytope, then its Pukhlikov–Khovanskii ring R_P is isomorphic to the cohomology ring of the toric variety X_P constructed using P (the proof is based on the fact that the cohomology ring $H^*(X_P)$ is generated by the degree two component). Consequently, the ring R_P models nicely the intersection theory on X_P: intersection product of cycles on X_P corresponds to the intersection of faces of P.

Similarly, to each regular compactification X_π one can assign a polytope $\tilde{P}_\pi$ of dimension n which fibers over the weight polytope P_π (this is a partial case of a more general construction by Kaveh and Khovanskii [19]). In the case $G = GL_N(\mathbb{C})$, the fiber over a weight $\lambda \in P_\pi$ is the product of two Gelfand-Zetlin polytopes Q_λ corresponding to the irreducible representation of G with the highest weight λ (in this case polytope $\tilde{P}_\pi$ was first constructed by Okounkov [25]). The polytope $\tilde{P}_\pi$ captures the geometry of X_π much better than the smaller polytope P_π (e.g. the Brion–Kazarnovskii formula becomes $H_\pi^n = n!\mathrm{vol}(\tilde{P}_\pi)$, which is completely analogous to the Koushnirenko formula in the torus case). However, in contrast with the torus case the polytope

$\tilde{P}_\pi$ is in general non-simple and the cohomology ring $H^*(X_\pi)$ is not generated by the degree two component. Still it might be possible to use the Pukhlikov–Khovanskii ring to describe $H^*(X_\pi)$. A crucial step is to understand the relation between Gelfand-Zetlin polytope Q_λ (for strictly dominant λ) and the cohomology ring of the flag variety G/B. This is what I am doing now.

Recall that there is a natural basis in $H^*(G/B)$ given by the Schubert cycles (which can be defined as the closures of B-orbits). First, I constructed a correspondence between Schubert cycles and some faces of the Gelfand-Zetlin polytope and using this correspondence proved the following interpretation of the Chevalley formula for the intersection product of a Schubert cycle with a divisor. Let H_λ be a divisor on G/B corresponding to the weight λ. For a face Γ of Q_λ denote by X_Γ the Schubert cycle corresponding to Γ. Then under some conditions on Γ (see [16, Theorem 5.5] for a more general statement that holds for all faces)

$$H_\lambda X_\Gamma = \sum_{\Delta\subset\Gamma} d(v,\Delta)X_\Delta,$$

where the sum is taken over the facets Δ of Γ that correspond to the Schubert cycles X_Δ of codimension one at the boundary of X_Γ. Here v is a fixed vertex of the face Γ and $d(v,\Delta)$ denotes the integral distance from v to the face Δ. Note that in this form the formula is completely analogous to the well-known formulas for toric varieties.

However, it is not always possible to represent a Schubert cycle by a single face whose combinatorics properly reflects geometry of the Schubert cycle. One reason for this is that the intersection of two Schubert cycles (unlike the intersection of the closures of torus orbits in a toric variety) is not necessarily a single Schubert cycle. There are several approaches to represent Schubert cycles by unions of faces [21, 22] but they do not give any flexibility (there is only one representation for each Schubert cycle). Together with Evgeny Smirnov and Vladlen Timorin we are currently investigating a new approach using the Pukhlikov-Khovanskii ring R_{Q_λ} of the Gelfand–Zetlin polytope Q_λ. We proved that the ring R_{Q_λ} is isomorphic to the cohomology ring of G/B and using this isomorphism expressed Schubert cycles by linear combinations of faces of Q_λ (since Q_λ is non-simple it required some extra work to show that the elements of R_{Q_λ} can still be represented by linear combinations of faces). In particular, we got the same representation as in [22] (which is formally similar to the Fomin–Kirillov theorem [12] describing all monomials in a given Schubert polynomial) and we hope to get many other interesting representations using relations between faces of Q_λ in the ring R_{Q_λ}. We also got a simple description of relations between facets: they are spanned by four-term relations (sum of two facets=sum of two facets), three-term relations (sum of two facets=one facet) and one two-term relation (one facet=another facet).

Our next goal is to model Schubert calculus on G/B by intersecting faces of the Gelfand–Zetlin polytope (using our approach it is possible to represent two Schubert cycles by unions of faces with transverse intersections). We hope to get an explicit combinatorial formula for the structure coefficients of $H^*(G/B)$ (in the basis of Schubert cycles) in terms of the Gelfand–Zetlin polytope so that the formula would imply the non-negativity of structure coefficients. The above interpretation of the Chevalley formula can be viewed as the first step in this direction.

Push-forward maps in algebraic cobordism. Let X be a smooth algebraic variety over a field of zero characteristic, and E a vector bundle of rank r over X. Denote by $Y := \mathbb{P}(E^*)$ the variety of hyperplanes in E, and by $\pi : Y \to X$ the natural projection. In the case $r = 2$, the Vishik–Quillen formula [26] for the push-forward $\pi_* : \Omega^*(Y) \to \Omega^*(X)$ has a nice interpretation in terms of generalized divided difference operators [13, Proposition 2.1]. Namely, using the projective bundle formula we can identify $\Omega^*(Y)$ with the quotient of the ring of formal power series $\Omega(X)^*[[y_1, y_2]]$ and on the latter ring we define the generalized divided difference operator A_1 by the formula

$$A_1 = (1 + \sigma_1)\frac{1}{y_1 -_F y_2},$$

where σ_1 is the operator permuting y_1 and y_2, and $-_F$ denotes subtraction with respect to the universal formal group law. My goal is to find an analogous interpretation of the push-forward π_* for arbitrary r. Such an interpretation in the Chow ring case is well-known, namely, π_* can be described using the composition $A_1 \cdots A_{r-1}$ of $r - 1$ usual divided difference operators. However, in the cobordism case extra terms do appear. I have a geometric argument using a double point relation which allows to describe these extra terms by induction on r.

Possible applications of a formula for push-forward in algebraic cobordism via divided difference operators might include translation of some of the topological results in [8, 7] to the algebro-geometric setting. In particular, this might lead to an explicit formula for the push-forward for the variety of complete flags in E.

Teaching experience and plans

I have been teaching mathematics for over 10 years. At the Stony Brook University and Jacobs University Bremen, I taught over 10 courses ranging from elementary and advanced undergraduate courses to graduate core and topics courses. The complete list of courses with programs and teaching materials is available on my homepage http://www.mccme.ru/ valya/. For two years, I had been teaching topics courses for high school and undergraduate students at the summer school “Contemporary Mathematics” in Dubna, Russia.

Right now I am teaching a topics course ”Geometry of spherical varieties” at the Independent University of Moscow. Using my notes for this course I plan to write a textbook, which can serve as an elementary introduction to geometry of spherical varieties. Preliminary version of the first few chapters (typed with help of Igor Netai) is available at the course webpage http://www.mccme.ru/ valya/spherical.html.

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