On Game Interpretations of Intuitionstic Logic

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Int — intuitionistic propositional logic.

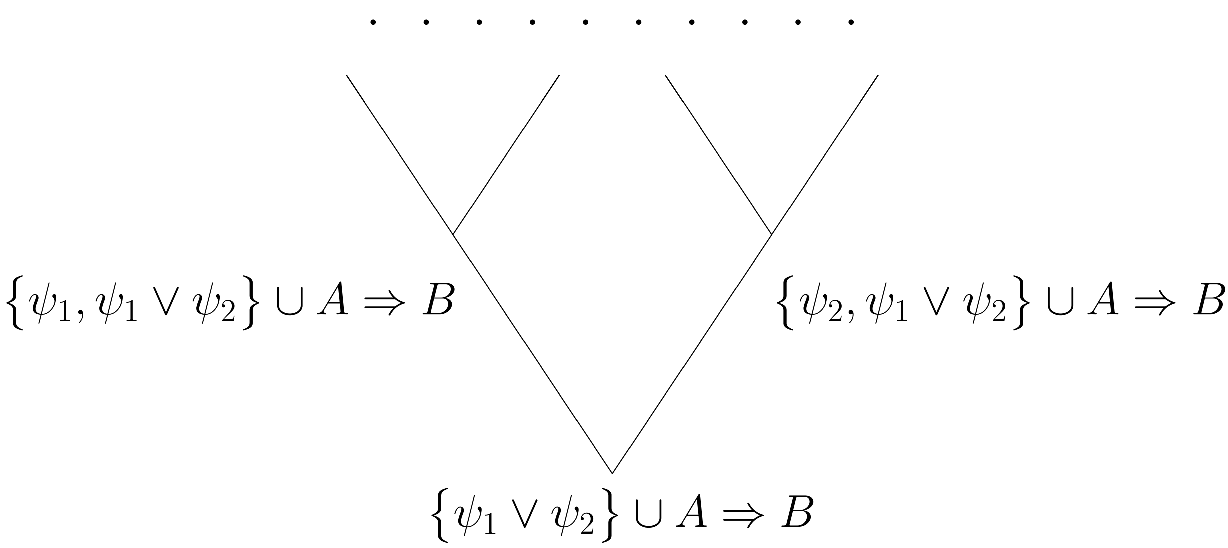
A calculus of sequences $A \Rightarrow B$,
where A and B are finite sets of propositional formulas.

A classical theorem.
$\vdash A \Rightarrow B$ *iff* $(\bigwedge A \to \bigwedge B) \in \mathrm{Int}$.

($\bigwedge A$ is the conjunction of all formulas from A, $\bigwedge \emptyset$ is the true formula $\top$.)

Any inference of sequence $A \Rightarrow B$ can be presented as a tree with sequences in its nodes and a $A \Rightarrow B$ in its root.

Example



Because of 'sub-formula' property the height of the inference tree is limited by a polynomial of the size of the 'root sequence'. So, a sequence is provable iff there is a winning strategy of the First Player in the following 'polynomial ' game.

The Player I tries to demonstrate the provability of the sequence $A \Rightarrow B$. He shows two new sequences $C_1 \Rightarrow D_1$, $C_2 \Rightarrow D_2$, pretending that they are provable and from which *in one step* the sequence $A \Rightarrow B$,is deducible and $A \subseteq C_i$, $B \subseteq D_i$, $i = 1,\ 2$.

The Player II tries to refute the provability of the sequence $A \Rightarrow B$.

He indicates one of the sequences $C_1 \Rightarrow D_1$, $C_2 \Rightarrow D_2$, pretending that it is not provable.

In case that the Player I rejects to make a move he wins iff formula $A \Rightarrow B$ is an axiom.

Evidently, we can check in polynomial time is a move correct and is a final position a winning one.

Proposition. The formula φ belongs to Int iff The Player I has a winning strategy in the described game for $\emptyset \Rightarrow \varphi$.

In September of 2005 at the International conference «Computer Science Applications of Modal Logic» in Moscow I. Mezhirov proposed a new game semantics for Int. In some aspect it is simpler that one we described, because all the positions of his game are sequences, not pairs of sequences.

We propose a new, more intuitive game semantics for Int having the same good property;

Game of «mutual respect»

Initial position — $\emptyset \Rightarrow \{\varphi\}$.

- The First Player's move — $\emptyset \Rightarrow B_1, \quad \varphi \in B_1$
  (pretending that B_1 — is the maximal set for which $\vdash \emptyset \Rightarrow B_1$).
- The II Player's move — $A_1 \Rightarrow B_1$
  (pretending that, A_1 — is a maximal set for which $\nvdash A_1 \Rightarrow B_1$.)

. . . .

- The First Player's move — $A_{n-1} \Rightarrow B_n, \quad B_{n-1} \subsetneq B_n$
  (pretending that, B_n — is the maximal set for which $\vdash A_{n-1} \Rightarrow B_n$).
- The Second Player's move — $A_n \Rightarrow B_n, \quad A_{n-1} \subsetneq A_n$.
  (pretending that, A_n — is a maximal set for which $\nvdash A_n \Rightarrow B_n$).

Rejection to move of one of the players means the end of the Game.

Lemma. *The set of sequences $A \Rightarrow B$, with a maximal set B can be separated in polynomial time from the set of not provable sequences $C \Rightarrow D$, where C is maximal.* $\square$

All known algorithms that separate sets of deducible and non deducible sequences use polynomial zone, but exponential time.

Theorem. *A propositional formula φ belongs to* Int *iff the First Player has a winning strategy in the Game of mutual respect for $\emptyset \Rightarrow \{\varphi\}$.* $\square$

I would like to thank the organizers for the opportunity to present my talk at this conference and congratulate Sergey Ivanovich with his 75th birthday.