We consider the following game between Mathematician and Adversary. A natural $n \geq 2$ is a parameter of the game. A game position is $n+1$ positive real numbers $L, L_1, \ldots, L_n$. Denote by $L(t), L_1(t), \ldots, L_n(t)$ their values after step t.

Before the game (at step 0) all these numbers are equal to zero.

At step t, Mathematician announces real numbers $p_1, \ldots, p_n \in [0,1]$ such that $\sum_i p_i = 1$. Then Adversary announces numbers $l_1, \ldots, l_n \in [0,1]$ (not necessarily summing up to 1). And then the position is updated:

$$L_i(t) = L_i(t-1) + l_i, \quad i = 1, \ldots, n\,,$$
$$L(t) = L(t-1) + p_1 l_1 + \cdots + p_n l_n\,.$$

The value $L(t) - \min_i L_i(t)$ is the loss of Mathematician (who tries to make it smaller) and the gain of Adversary.

Theorem 1. *For $n = 2$, for any T, Adversary has a polynomially computable strategy such that at each step either $l_1 = 1$, $l_2 = 0$ or $l_1 = 0$, $l_2 = 1$, and this strategy guarantees that*

$$L(T) - \min_i L_i(T) \geq c\sqrt{T}\,,$$

where c is a constant.

This strategy can be considered as a strategy against Learner in the absolute loss game or randomized simple prediction game (cf. [1]).

Proof. Let $\alpha < 1$ be a positive constant that will be specified later.

For each $i = 1, 2$, the strategy stores the number $g_i(t) = \sqrt{T} - L(t) + L_i(t)$.

At step $t < T$ the strategy does the following. If $g_i(t) < \sqrt{\alpha T}$ for one of i, then the strategy takes $l_i = 0$. Otherwise, the strategy computes $g_1(t{+}1)g_2(t{+}1)$ for both possible moves of Adversary (the move $(p, 1-p)$ of Mathematician is known at the moment), and chooses the move that minimizes this product. In other words, the move of Adversary is $l_1 = 0$, $l_2 = 1$ if $p(g_1(t){+}g_2(t)){-}g_2(t) < 0$, and $l_1 = 1$, $l_2 = 0$ otherwise.

If $g_i(t) < \sqrt{\alpha T}$, the strategy's move guarantees that $g_i(t+1) \leq g_i(t)$, thus $g_i(T) < \sqrt{\alpha T}$ and $L(t) - L_i(t) > (1 - \sqrt{\alpha})\sqrt{T}$.

Let us prove that it will happen at some step $t \leq T$ that $g_i(t) < \sqrt{\alpha T}$ for one of i. It suffices to prove that $g_1(T)g_2(T) < \alpha T$.

Let us estimate the change of $g_1(t)g_2(t)$ at one step assuming that $g_i(t) \geq \sqrt{\alpha T}$ for $i = 1, 2$. Let the move of Mathematician be $(p, 1-p)$. Then $g_1(t+1)g_2(t+1)$ can be $(g_1(t) - (1-p))(g_2(t) + p)$ or $(g_1(t) + (1-p))(g_2(t) - p)$ depending on the move of Adversary. The minimum of these two values is $g_1(t)g_2(t) - |pg_1(t) - (1-p)g_2(t)| - p(1-p)$. It is easy to see that the minimum of $p(1{-}p){+}|p(g_1(t){+}g_2(t)){-}g_2(t)|$ over p is attained at $p = g_2(t)/(g_1(t){+}g_2(t))$ (we assume here that $g_1(t) + g_2(t) \geq 1$, which holds if $2\sqrt{\alpha T} \geq 1$), thus the strategy guarantees that

$$g_1(t+1)g_2(t+1) \leq g_1(t)g_2(t) - g_1(t)g_2(t)/(g_1(t) + g_2(t))^2$$

independent of the move of Mathematician.

Let us bound $g_1(t) + g_2(t)$ from above. We see that $g_1(t)g_2(t)$ does not increase, therefore $g_1(t)g_2(t) \leq g_1(0)g_2(0) = T$ and $g_1(t) + g_2(t) \leq g_1(t) +$

$T/g_1(t)$. Without loss of generality, assume that $g_1(t) \leq g_2(t)$, then $g_1(t) \leq \sqrt{T}$, and the maximal (over $g_1(t) \geq \sqrt{\alpha T}$) value of $g_1(t) + T/g_1(t)$ is attained at $g_1(t) = \sqrt{\alpha T}$. Therefore, we get $g_1(t) + g_2(t) \leq \sqrt{T}(\sqrt{\alpha} + 1/\sqrt{\alpha})$, and thus

$$g_1(t+1)g_2(t+1) \leq g_1(t)g_2(t)\left(1 - \frac{\alpha}{(1+\alpha)^2 T}\right).$$

We have

$$g_1(T)g_2(T) \leq T\left(1 - \frac{\alpha}{(1+\alpha)^2 T}\right)^T \leq T\mathrm{e}^{-\frac{\alpha}{(1+\alpha)^2}},$$

and it suffices to choose α such that

$$\mathrm{e}^{-\frac{\alpha}{(1+\alpha)^2}} < \alpha.$$

It is easy to check that $\alpha = \mathrm{e}^{-0.16}$ works, and then $c = 1 - \sqrt{\alpha}$ is between 0.07 and 0.08. (This value of α is not optimal, but in any case $\alpha > \mathrm{e}^{-0.25}$.) □

References

[1] N. Cesa-Bianchi, Y. Freund, D. Haussler, D. Helmbold, R. Shapire, M. Warmuth. How to Use Expert Advice. *JACM*, 44(3):427–485, 1997.