LOWER LIMITS OF FREQUENCIES IN COMPUTABLE SEQUENCES AND RELATIVIZED A PRIORI PROBABILITY*

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(*Translated by Brie Ellis*)

For each computable sequence of natural numbers one can define a measure on $\mathbf{N}=\{1,2,\cdots\}$ by taking the measure of a natural number to be the lower limit of its frequency in the beginning segments of a selected sequence. This paper establishes that among the measures so determined there is a maximal one (to within a multiplicative constant) and that it coincides with the a priori probability in the sense of [1] relative to a universal denumerable set $\mathbf{0}'$ (concerning relativization see [2, § 9.2]).

Suppose we have a computable sequence of natural numbers $f(0), f(1), \cdots$ and an arbitrary natural number x. Consider the sequence whose nth term is the frequency of occurrence of x among the first n terms of the sequence f, i.e., the number of $k<n$ such that $f(k)=x$, divided by n. For each x we consider the lower limit of this sequence, which we call the lower frequency of x in the sequence of f and write it as $\mathrm{Freq}_f(x)$.

It is easily verified that the sum of all the $\mathrm{Freq}_f(x)$ over all $x\in\mathbf{N}$ does not exceed 1. We assign to each computable sequence $f(0), f(1), \cdots$ of natural numbers the measure defined on subsets of the natural numbers by considering the measure of a singleton set $\{x\}$ to be $\mathrm{Freq}_f(x)$. The theorem below shows that among all such measures there is a maximal one (to within a multiplicative constant) and it establishes its connection with the a priori probability.

We recall that the a priori probability is the greatest non-negative function $p:\mathbf{N}\to\mathbf{R}^1$, to within a multiplicative constant, for which the set $\{\langle r, x\rangle \mid r\in\mathbf{Q}, x\in\mathbf{N}, r<p(x)\}$ is denumerable ($\mathbf{Q}$ is the set of rational numbers). The existence of such a function is proved in [1]. This proof remains valid if "dunumerable" is replaced in the definition of a priori probability by "dunumerable relative to $\mathbf{0}'$." (Sets are said to be denumerable relative to $\mathbf{0}'$ [2, §§ 9.2-3] if they are the range of a function computable by an algorithm with an oracle for some denumerable set. An "algorithm with an oracle for a set X" is an algorithm which can be subjected to a procedure anwering the question "Is a in X?" for any a.) Replacing "denumerability" by "denumerability relative to $\mathbf{0}'$" in the definition of a priori probability, we arrive at the notion of relativized a priori probability relative to $\mathbf{0}'$, and it is the one we shall use.

THEOREM. (A) *There is a computable sequence f such that for any computable sequence g, some $C>0$ and all $x\in\mathbf{N}$,*

$$\mathrm{Freq}_f(x)\geqq C\,\mathrm{Freq}_g(x)$$

(B) *For this sequence f there are constants $C_1, C_2>0$, such that*

$$C_1 p(x)\geqq \mathrm{Freq}_f(x)\geqq C_2 p(x),$$

where p is the relativized a priori probability relative to $\mathbf{0}'$.

Proof. It suffices to establish two facts:

(1) for every computable sequence f the function $x\mapsto\mathrm{Freq}_f(x)$ is denumerable relative to $\mathbf{0}'$ (this means that the set $\{\langle r, x\rangle \mid r\in\mathbf{Q}, x\in\mathbf{N}, r<f(x)\}$ is denumerable relative to $\mathbf{0}'$);

(2) for every function $p\geqq 0$ that is denumerable from below relative to $\mathbf{0}'$ and for which $\sum_x p(x)\leqq 1$, there exists a computable sequence f such that $p(x)\leqq\mathrm{Freq}_f(x)$ for any $x\in\mathbf{N}$.

The first fact is easily established. It suffices to observe that the property $r<\mathrm{Freq}_f(x)$ is equivalent to: "there exists an N such that for all $k>N$ the fraction of x in the beginning segment $f(0), \cdots, f(k-1)$ exceeds r," and this assertion has the form $\exists N\,\forall k R(r, x, N, k)$, where R is a decidable predicate and so yields a set that is denumerable relative to $\mathbf{0}'$.

To prove the second fact we need a lemma. By a simple semidistribution on $\mathbf{N}$ we shall mean a function $r:\mathbf{N}\to\mathbf{Q}$ with non-negative values, finitely nonzero, and such that $\sum_x r(x)\leqq 1$.

LEMMA. *Let r_k be a computable sequence of semidistributions. Then there exists a computable sequence of natural numbers $f(0), f(1), \cdots$, such that*

$$\mathrm{Freq}_f(x)\geqq\liminf_k r_k(x).$$

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Proof. For each k we construct a finite sequence α_k of natural numbers such that the frequency of occurrence of x (denote it by $r'_k(x)$) is greater than or equal to $r_k(x)$ for every $x \in \mathbf{N}$. The sequence f will have the form

$$\alpha_0 \cdots \alpha_0 \alpha_1 \cdots \alpha_1 \alpha_2 \cdots \alpha_2 \cdots,$$

where α_k is repeated n_k times, $k = 0, 1, \cdots$; n_k is chosen to be so large that the addition to $\alpha_k \cdots \alpha_k$ of any sequence of natural numbers of length at most $|\alpha_{k+1}| + n_0|\alpha_0| + \cdots + n_{k-1}|\alpha_{k-1}|$ ($|\alpha|$ is the length of the sequence α) changes the frequency by little (by at most $1/k$).

Consider an arbitrary beginning segment of the sequence constructed. It has the form

$$\alpha_0 \cdots \alpha_0 \alpha_1 \cdots \alpha_1 \cdots \alpha_{k-1} \cdots \alpha_{k-1} \beta,$$

where β is some start of the sequence α_k. We form two groups out of the natural numbers appearing in this segment: one containing the numbers in $\alpha_0 \cdots \alpha_0 \cdots \alpha_{k-1} \cdots \alpha_{k-1}\beta$, and the other the numbers in $\alpha_k \cdots \alpha_k$. In the first group the frequencies are close to r'_{k-1} (to within $1/(k-1)$) and in the second group they are equal to r'_k. Therefore the frequencies over the whole beginning segment occupy (to within $1/(k-1)$) a sort of average position between r'_{k-1} and r_k. Hence the assertion in the lemma is true.

Let us turn to the proof of the theorem. Let p be a non-negative function from $\mathbf{N}$ into $\mathbf{R}^1$ that is denumerable from below relative to $\mathbf{0}'$. Such a function can be represented as the limit of an increasing $\mathbf{0}'$-computable sequence of simple semidistributions u_k (e.g., we can define $u_k(x)$ to be 0 if $x \geqq k$ and $u_k(x)$ to be the largest rational number r for which the pair $\langle r, x\rangle$ occurs in k steps of the $\mathbf{0}'$-enumeration of the set $\{\langle r, x\rangle \mid r < p(x)\}$ if $x < k$). Every $\mathbf{0}'$-computable function is the limit of a stabilized computable sequence: $u_k = \lim_s u_{ks}$, where u_{ks} is a simple semidistribution depending computably on k and s, and among all the u_{ks} for a given k there are only a finite number of distinct ones (see [3, chapter 6]). We now construct a sequence of simple semidistributions to which the lemma is applied. For each s, consider simple semidistributions $u_{1s}, u_{2s}, \cdots, u_{ss}$ and choose from among them an increasing beginning segment of maximal length (for which $u_{1s}(x) \leqq \cdots \leqq u_{ts}(x)$ for any x). Take r_s to be the last term u_{ts}.

To complete the proof it remains to show that if $r < p(x)$, then $r < r_s(x)$ for all s except a finite number. Indeed, if $r < p(x)$, then $r < u_k(x)$ for some k. We look at $u_{1s}, \cdots, u_{ks}$ as s increases. For s sufficiently large, they will be equal to $u_1, \cdots, u_s$. For such s (one can even assume $s > k$) the maximal increasing segment will contain $u_{1s}, \cdots, u_{ks}$ (since the sequence u_l increases) and consequently $r_s(x) \geqq u_{uks}(x) > r$.

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ON THE OPERATING TIME OF ERRORLESS PROBABILISTIC TURING MACHINES*

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Probabilistic Turing machines differ from deterministic Turing machines (see the definition in [1]) only in that at each stage of operation probabilistic machines can use the output of a random number generator which puts out the values {0, 1} equiprobably and independently of the output at other times.

The following definition of language recognition on a probabilistic machine in time $t(x)$ with a probability p is used. It is required that for any input word x, the following event occurs with a probability at least p (where p is a fixed number $>\frac{1}{2}$): the machine stops in at most time $t(x)$ and gives the right result. In particular, if x belongs to this language then the result "belongs" is put out with a probability $\geqq p > \frac{1}{2}$ (and also in at most $t(x)$ steps), while the result "does not

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