Logical operations and Kolmogorov complexity

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Kolmogorov complexity $K(x)$ of a binary string x is defined as minimal length of a program that generates this string. This definition can be extended to sets of strings. Let A be a (finite or infinite) set of strings. We define the complexity $K(A)$ as the length of a shortest program that generates some string $x \in A$. Informally, we consider A as a problem "Generate any element of A"; $K(A)$ is complexity of this problem. Evidently, $K(A) = \min\{K(x) \mid x \in A\}$, so this generalization gives nothing really new.

However, it can be combined with the definition of logical operations on sets of strings that goes back to Kolmogorov's paper on meaning of intuitionistic logic and Kleene's notion of realizability. Let A and B be two sets of strings. We define sets $A \wedge B$, $A \vee B$ and $A \to B$ as follows:

- $A \wedge B = \{\langle a, b\rangle \mid a \in A, b \in B\}$
- $A \vee B = \{\langle 0, a\rangle \mid a \in A\} \cup \{\langle 1, b\rangle \mid b \in B\}$
- $A \to B = \{p \mid [p](x) \in B \text{ for all } x \in A\}$

Here $[p](x)$ stands for the output of p (considered as a program) when applied to input x.

Let $A(p_1, \ldots, p_k)$ be a propositional formula with connectives $\vee, \wedge, \to$. Substitute singletons $\{a_1\}, \ldots, \{a_k\}$ for variables and denote the resulting set by $A(a_1, \ldots, a_k)$. For many formulas $A(p_1, \ldots, p_k)$ the complexity of $A(a_1, \ldots, a_k)$ may be expressed in terms of of the complexity of $a_1, \ldots, a_k$, their pairs, triples etc. up to an additive term $O(\log K(a_1, \ldots, a_k))$. Our first result is that this is not always the case.

Theorem 1. *The complexity of the problem $(\{a\} \to \{c\}) \wedge (\{b\} \to \{d\})$ is not expressible in terms of complexity of a, b, c, d, their pairs, triples and the quadruple up to an additive $o(K(a, b, c, d))$ term.*

The goal of Kolmogorov and Kleene was to provide an interpretation of the intuitionistic propositional calculus. Following this idea, denote by $L_{O(1)}$ $[L_{\log}]$ the set of formulas $A(p, q, \ldots)$ such that $K(A(x, y, \ldots)) = O(1)$ $[K(A(x, y, \ldots)) = O(\log K(x, y, \ldots))]$ for any strings $x, y, \ldots$.

Theorem 2. *The following inclusions hold: $IPC \subset L_{O(1)} \subset L_{\log} \subset CPC$ (here IPC [CPC] stands for the set of formulas provable in the intuitionistic [classical] propositional calculus). The inclusions $IPC \subset L_{\log}$ and $L_{\log} \subset CPC$ are proper.*

From the logical point of view the set $L_{\log}$ is not very interesting as it is not closed under substitution. The same applies to the set $L_{O(1)}$: it is unknown whether this set is closed under substitution. So consider the sets of formulas $\bar{L}_{O(1)}$ and $\bar{L}_{\log}$, which are defined similar to $L_{O(1)}$ and $L_{\log}$ but we substitute arbitrary sets (not only singletons) for variables. Both sets $\bar{L}_{O(1)}$ and $\bar{L}_{\log}$ are superintuitionistic logics.

Question. Are the inclusions IPC $\subset \bar{L}_{O(1)} \subset \bar{L}_{\log}$ proper?

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